ON THE LIMIT THEOREMS FOR CONVOLUTIONS OF POWER SERIES TYPE

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Abstract: *The lifetime represented as the random sum of the random variables (r.v.) is found in many issues related to Reliability, Actuarial, Queuing Theory, Renewal Theory and so one. In our work we intend to research the lifetime distribution when the random number of r.v. has a power series distribution, abbreviated as PSD, the components of the sum being nonnegative, independent and identically distributed random variables (i.i.d.r.v.), of (absolutely) continuous type. Results are expressed through Laplace transform. These results are used, as an illustration, for the demonstration / validation of limit theorems for power series type's convolution.*

Keywords: *convolution, Laplace transform, limit theorems, PSD.*

1. INTRODUCTION

By convolutions of power series (PSD = "power series distribution") we understand the lifetime distribution represented by means of sum of non negative i.i.d. r.v., considered in a random number with distribution of PSD type. They generates , in particular, geometric or Pascal convolutins, which appear in some limit theorems in Reliability Theory ([1], [2], [8] and [10]). Such convolutions also appear in problems related to the Queueing Theory, Acturial Mathematics , Risc Theory or Renewal Processes. Distributions of PSD type which enclose a whole class of discrete distributions it was introduced in the papper [5].

The notion of "the power series distribution " class is due to Noack [11] and Kosambi [6]; Noack gives a particular importance to the discrete distributions which belong to this class (for example: binomial, Poisson, logarithmic, geometric, negative binomial [5]).

In the following we reproduce the definition of the class of power series distributions.

Let's consider r.v. Z such that $P(Z \in \{1,2,...\}) = 1$.

Definition 1.1. We say that v.a. Z has a power series distribution (PSD), if:

$$P(Z = z) = \frac{a_z \theta^z}{A(\theta)},\ z = 1,2,... \tag{1}$$

where $\theta \in (0,\tau)$, $\tau > 0$, $a_1, a_2,...$ are positive real numbers and τ is a positive number meaning the convergence radius of power series (*series function*),

$$A(\theta) = \sum_{z \geq 1} a_z \theta^z,\ \forall \theta \in (0,\tau),$$

and θ is *the power parameter* of the distribution.

Consequence 1.1. To the PSD class belong, in particular, the following zero truncated distributions: binomial, Poisson, logarithmic, geometric, Pascal and negative binomial. Their representation is shown in the Table 1.1.

Consequently, we present for each distribution, the representative elements of the PSD class: sequence $(a_z)_{z \geq 1}$, series function $A(\theta)$, as well as the connection between the paramether θ of the power series and the distribution parameters which belong to this class;

all these results are centralized in Table 1.1.

Table 1. The representative elements of PSD class

Distribution	a_z	θ	$A(\theta)$	τ
$Binom^*(n,p)$	$\binom{n}{z}$	$\frac{p}{1-p}$	$(1+\theta)^n-1$	∞
$Poisson^*(\lambda)$	$\frac{1}{z!}$	λ	$e^\theta-1$	∞
$Log(p)$	$\frac{1}{z}$	p	$-\ln(1-\theta)$	1
$Geom^*(p)$	1	$1-p$	$\frac{\theta}{1-\theta}$	1
$Pascal(k,p)$	$\binom{z-1}{k-1}$	$1-p$	$\left(\frac{\theta}{1-\theta}\right)^k$	1
$BN^*(k,p)$	$\binom{z+k-1}{z}$	p	$(1-\theta)^{-k}-1$	1

2. LIMIT THEOREMS ABOUT CONVOLUTIONS OF POWER SERIES TYPE

Given the geometric convolution of the exponential distribution and considering the elements characteristic to the PSD class for the geometric distribution, we obtain a new variant of Brown's limit theorem ([1], [2]):

Theorem 2.1. ([9]) If $X_i, i \geq 1$ are a non negative i.i.d.r.v., with the Laplace transform of p.d.f. $\varphi_{X_i}(s) \equiv \varphi(s)$, such that exists mean value $EX_i = -\varphi'(s)\Big|_{s=0} = \frac{1}{\lambda}$, $\lambda>0$ and

$N \sim Geom^*(p)$, 0< p<1, being independent of r.v. X_i, $i \geq 1$, then $\lambda p Y_N \underset{p\to 0}{\Rightarrow} Exp(1)$ for $Y_N = X_1 + X_2 + ... + X_N$.

Proof. Assume that $N \sim Geom^*(p)$, $p \in (0,1)$, i.e., N belong to the PSD, with $A(\theta) = \frac{\theta}{1-\theta}$, $\theta \in (0,1)$, $\theta = 1-p$. Then the Laplace transform of p.d.f. for r.v. Y_N is given by,

$$\varphi_{Y_N}(s) = \frac{A(\theta\varphi(s))}{A(\theta)} = \frac{p\varphi(s)}{1-(1-p)\varphi(s)}$$

$p \in (0,1)$.

Consequently the Laplace transform of v.a. $\lambda p Y_N$, is characterised by the following relation:

$$\varphi_{\lambda p Y_N}(s) = \varphi_{Y_N}(\lambda p s) = \frac{p\varphi(\lambda p s)}{1-(1-p)\varphi(\lambda p s)},$$

$p \in (0,1)$, $\lambda > 0$.

Applying the rule of l 'Hospital and taking into account the properties of Laplace transform, we find that:

$$\lim_{p\to 0}\varphi_{\lambda p Y_N}(s) = \lim_{p\to 0}\frac{\varphi(\lambda p s) + \lambda p s\varphi'(\lambda p s)}{\varphi(\lambda p s) - (1-p)\lambda s\varphi'(\lambda p s)} = \frac{1}{1+s},$$

i.e., $\lambda p Y_N \underset{p\to 0}{\Rightarrow} Exp(1)$, that's complet the proof.

The following result presents a new variant of the limit theorem (generalization of Brown's limit theorem in [8] and [10]).

Theorem 2.2. ([9]) If $X_i, i \geq 1$ are a non negative i.i.d.r.v., with Laplace transform of p.d.f. $\varphi_{X_i}(s) \equiv \varphi(s)$, such that exists mean value $EX_i = -\varphi'(s)\Big|_{s=0} = \frac{1}{\lambda}$, $\lambda>0$ and

$N \sim Pascal(k,p)$, $p \in (0,1)$, being independent of r.v. X_i, $i \geq 1$, then $\lambda p Y_N \underset{p\to 0}{\Rightarrow} Erlang(k,1)$

for $Y_N = X_1 + X_2 + ... + X_N$.

Proof. Assume that $N \sim Pascal(k, p)$, $p \in (0,1)$, i.e., N belong to the PSD, with $k \in N^*$ and

$$A(\theta) = \left(\frac{\theta}{1-\theta}\right)^k, \ \theta \in (0,1), \ \theta = 1 - p.$$

The Laplace transform of p.d.f of r.v. Y_N is :

$$\varphi_{Y_N}(s) = \frac{p^k \varphi^k(s)}{[1-(1-p)\varphi(s)]^k}.$$

Consequently, ,

$$\varphi_{\lambda p Y_N}(s) = \varphi_{Y_N}(\lambda p s) = \frac{p^k \varphi^k(\lambda p s)}{[1-(1-p)\varphi(\lambda p s)]^k}$$

$p \in (0,1)$, $\lambda > 0$.

Applying the l'Hospital's rule, we obtain that: $\lim\limits_{p \to 0} \varphi_{\lambda p Y_N}(s) = \frac{1}{(1+s)^k}$, $k \in N^*$, i.e., the limits is exactly the Laplace transform of p.d.f.

$$f_{Y_N}(t) = \frac{t^{k-1}}{(k-1)!} e^{-t}$$

for Erlang distribution with k degrees of freedom and parameter 1. So, $\lambda p Y_N \underset{p \to 0}{\Rightarrow} Erlang(k,1)$

that's complet the proof.

Remark 2.1. We observe that for non negative r.v. X_i, $i \geq 1$ the condition to be governed by the strong law of large numbers (according to theorems in [1] and [10]) becomes unnecesary in the both Theorem 2.1 and Theorem 2.2.

Now we need some auxiliar results, where by $Geom^*(p)$ we understand the geometric distribution truncated to zero.

Lemma 2.1. ([4]) If $N \sim Geom(p)$, $p \in (0,1)$, then $pN \Rightarrow Exp(1)$ for $p \to 0$.

Proposition 2.1 ([7,8]) If $N \sim Pascal(k, p)$, $k \in N^*, p \in (0,1)$, then $pN \underset{p \to 0}{\Rightarrow} Z \sim Erlang(k,1)$

Proposition 2.2. If r.v. $N \sim Geom^*(p)$, $p \in (0,1)$, r.v X_i, $i \geq 1$ are i.i.d. $Geom^*(p^*)$, $p^* \in (0,1)$, N and $(X_i)_{i \geq 1}$ being independent, then r.v. $Y_N \sim Geom^*(p\, p^*)$.

Proposition 2.3. If r.v. $N \sim Pascal(k, p)$, $p \in (0,1)$, $k \in N^*$, r.v. X_i, $i \geq 1$ are i.i.d. $Geom^*(p^*)$, $p^* \in (0,1)$, N and $(X_i)_{i \geq 1}$ being independent, then r.v. $Y_N \sim Pascal(k, p\, p^*)$.

As a consequense of Lema 2.1. we have the following proposition:

Proposition 2.4. In the conditions of Proposition 2.2. we have that

$$p\, p^* Y_N \underset{p \to 0}{\Rightarrow} Exp(1)$$

Similarly, we have

Proposition 2.5. In the conditions of Proposition 2.3. we have that

$$p\, p^* Y_N \underset{p \to 0}{\Rightarrow} Erlang(k,1)$$

In terms of the behaviour of parameter which define distribution r.v. N ∈ PSD, we can also formulate the following limit theorem:

Theorem 2.3. If v.a. N ∈ PSD, X_i, $i \geq 1$ are i.i.d.r.v., N and $(X_i)_{i \geq 1}$ being independent, then r.v. $Y_N = X_1 + X_2 + ... + X_N$ converges in distribution towards r.v. X_1, as well as:

(a) $N \sim Binom^*(n, p)$, $n = 1$ or $p \to 0$;

(b) $N \sim Poisson^*(\lambda)$, $\lambda > 0$, $\lambda \to 0$;

(c) $N \sim Log(p)$, $p \in (0,1)$, $p \to 1$;

(d) $N \sim Pascal(k, p)$, $p \in (0,1)$, $k \in N^*$, $p \to 1$.

Proof. The theorem results immediately, observing the that in each conditions (a)-(d) the probability $P(N = 1) \to 1$, and $P(N > 1) \to 0$.

The following two limit theorems are directly concerned with the connection among different distributions of r.v. $N \in$ PSD.

Theorem 2.4. Binomial convolution with parameters $n \in N^*$, $p \in (0,1)$ converges in distribution towards Poisson convolution with parameter $\lambda > 0$, as wel as $n \to \infty$, $p \to 0$ such that $np \to \lambda$.

Proof. According to the clasical Poisson theorem [3], r.v. $N \sim \mathrm{Binom}(n,p)$ converges in distribution towards Poisson distribution (λ), $\lambda > 0$ if $n \to \infty$, $p \to 0$ such that $np \to \lambda$. This imply our theorem.

Theorem 2.5. Negative binomial convolution with parameters $p \in (0,1)$ and $k \in N^*$ converges in distribution to the Poisson convolution parameter $\lambda > 0$ as well as $k \to \infty$, $p \to 1$ such that $k(1-p) \to \lambda$.

Proof. On the base of Feller's well known result [3], binomial negative distribution (non truncated) converges slightly towards Poisson distribution (non truncated). Since this property is also preserved when the corresponding distributions are truncated to zero, we obtain our statement of our theorem.

3. CONCLUSIONS

Because lifetime often occurs not only in Reliability, but also in Queueing Theory, Actuary, etc., a whole class of convolutions has been investigated, namely, those of power series type (in absolutely continuous). As a consequence of this approach a variant of the Limit Theorem has been obtained. This generalizes Brown's Theorem in which the restrictive condition that the sum of the r.v. be governed by the Strong Law of Large Numbers, is not necessary. New limits theorems in terms of convolution, are presented too.

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