AN INTRODUCTION TO THE DSM THEORY FOR THE COMBINATION OF PARADOXICAL, UNCERTAIN AND IMPRECISE SOURCES OF INFORMATION

Florentin SMARANDACHE*, **Jean DEZERT****

*Departament of Mathematics, University of New Mexico, Gallup, USA,
**ONERA/DTIM/IED, Châtillon, France

Abstract: *The management and combination of uncertain, imprecise, fuzzy and even paradoxical or high conflicting sources of nformation has always been, and still remains today, of primal importance for the development of reliable modern information systems involving artificial reasoning. In this introduction, we present a survey of our recent theory of plausible and paradoxical reasoning, known as Dezert-Smarandache Theory (DSmT) in the literature, developed for dealing with imprecise, uncertain and paradoxical sources of information. We focus our presentation here rather on the foundations of DSmT, and on the two important new rules of combination, than on browsing specific applications of DSmT available in literature. Several simple examples are given throughout the presentation to show the efficiency and the generality of this new approach.*

Keywords: *Dezert-Smarandache Theory, DSmT, Data Fusion, Plausible and Paradoxical Reasoning, Artificial Intelligence.*

1. INTRODUCTION

The management and combination of uncertain, imprecise, fuzzy and even paradoxical or high conflicting sources of information has always been, and still remains today, of primal importance for the development of reliable modern information systems involving artificial reasoning. The combination (fusion) of information arises in many fields of applications nowadays (especially in defense, medicine, finance, geo-science, economy, etc).

When several sensors, observers or experts have to be combined together to solve a problem, or if one wants to update our current estimation of solutions for a given problem with some new information available, we need powerful and solid mathematical tools for the fusion, specially when the information one has to deal with is imprecise and uncertain. In this paper, we present a survey of our recent theory of plausible and paradoxical reasoning, known as Dezert-Smarandache Theory (DSmT) in the literature, developed for dealing with imprecise, uncertain and paradoxical sources of information. Recent publications have shown the interest and the ability of DSmT to solve problems where other approaches fail, especially when conflict between sources becomes high.

We focus our presentation here rather on the foundations of DSmT, and on the two important new rules of combination, than on browsing specific applications of DSmT available in literature.

A particular attention is given to general (hybrid) rule of combination which deals with any model for fusion problems, depending on the nature of elements or hypotheses involved into them. The Shafer's model on which is based the Dempster-Shafer Theory (DST) appears only as a specific DSm hybrid model and can be easily handled by our approach as well. Several simple examples are given throughout the presentation to show the efficiency and the generality of this new approach.

2. FOUNDATIONS OF THE DSMT

The development of the DSmT (Dezert-Smarandache Theory of plausible and paradoxical reasoning [24, 6]) arises from the necessity to overcome the inherent limitations of the DST (Dempster-Shafer Theory [18]) which are closely related with the acceptance of Shafer's model for the fusion problem under consideration (i.e. the frame of *discernment* Θ defined as a finite set of *exhaustive* and *exclusive* hypotheses θ_i, i = 1, . . . , *n*), the third middle excluded principle (i.e. the existence of the complement for any elements/propositions belonging to the power set of Θ), and the acceptance of Dempter's rule of combination (involving normalization) as the framework for the combination of independent sources of evidence. Discussions on limitations of DST and presentation of some alternative rules to the Dempster's rule of combination can be found in [38, 39, 40, 34, 41, 8, 35, 15, 28, 32, 10, 14, 12, 17, 13, 24] and therefore they will be not reported in details in this paper. We argue that these three fundamental conditions of the DST can be removed and another new mathematical approach for combination of evidence is possible.

The basis of the DSmT is the refutation of the principle of the third excluded middle and Shafer's model, since for a wide class of fusion problems the intrinsic nature of hypotheses can be only vague and imprecise in such a way that precise refinement is just impossible to obtain in reality so that the exclusive elements θ_i cannot be properly identified and precisely separated. Many problems involving fuzzy continuous and relative concepts described in natural language and having no absolute interpretation like tallness/smallness, pleasure/pain, cold/hot, Sorites paradoxes, etc, enter in this category. DSmT starts with the notion of *free DSm model*, denoted $\mathcal{M}^f(\Theta)$, and considers Θ only as a frame of exhaustive elements θ_i, i = 1, . . . , n which can potentially overlap. This model is *free* because no other assumption is done on the hypotheses, but the weak exhaustivity constraint which can always been satisfied according the closure principle explained in [24]. No other constraint is involved in the free DSm model. When the free DSm model holds, the classic commutative and associative DSm rule of combination (corresponding to the conjunctive consensus defined on the free Dedekind's lattice) is performed.

Depending on the intrinsic nature of the elements of the fusion problem under consideration, it can however happen that the free model does not fit the reality because some subsets of Θ can contain elements known to be truly exclusive but also truly non existing at all at a given time (specially when working on dynamic fusion problem where the frame Θ varies with time with the revision of the knowledge available).

These integrity constraints are then explicitly and formally introduced into the free DSm model $\mathcal{M}^f(\Theta)$ in order to adapt it properly to fit as close as possible with the reality and permit to construct a *hybrid DSm model* $\mathcal{M}$ (Θ) on which the combination will be efficiently performed. Shafer's model, denoted $\mathcal{M}^0(\Theta)$, corresponds to a very specific hybrid DSm model including all possible exclusivity constraints. The DST has been developed for working only with $\mathcal{M}^f(\Theta)$ while the DSmT has been developed for working with any kind of hybrid model (including Shafer's model and the free DSm model), to manage as efficiently and precisely as possible imprecise, uncertain and potentially high conflicting sources of evidence while keeping in mind the possible dynamicity of the information fusion problematic. The foundations of the DSmT are therefore totally different from those of all existing approaches managing uncertainties, imprecisions and conflicts. DSmT provides a new interesting way to attack the information fusion problematic with a general framework in order to cover a wide variety of problems.

DSmT refutes also the idea that sources of evidence provide their beliefs with the same absolute interpretation of elements of the same frame Θ and the conflict between sources arises not only because of the possible unreliabilty of sources, but also because of possible different and relative interpretation of Θ, e.g. what is considered as good for somebody can be considered as bad for somebody else. There is some unavoidable subjectivity in the belief assignments provided by the sources of evidence, otherwise it would mean that all bodies of evidence have a same objective and universal interpretation (or measure) of the phenomena under consideration, which unfortunately rarely occurs in reality, but when bba are based on some *objective probabilities* transformations. But in this last case, probability theory can handle properly and efficiently the information, and the DST, as well as the DSmT, becomes useless. If we now get out of the probabilistic background argumentation for the construction of bba, we claim that in most of cases, the sources of evidence provide their beliefs about elements of the frame of the fusion problem only based on their own limited knowledge and experience without reference to the (inaccessible) absolute truth of the space of possibilities. First applications of DSmT for target tracking, satellite surveillance, situation analysis and sensor allocation optimization can be found in [24].

2.1. NOTION OF HYPER-POWER SET D^{Θ}

One of the cornerstones of the DSmT is the free Dedekind lattice [3] denoted *hyper-power set* in the DSmT framework. Let $\Theta = \{\theta_1, \dots, \theta_n\}$ be a finite set (called frame) of n exhaustive elements. The hyper-power set D^{Θ} is defined as the set of all composite propositions built from elements of Θ with $\cup$ and $\cap$ operators such that:

1. $\emptyset, \theta_1, \dots, \theta_n \in D^{\Theta}$.
2. If $A,B \in D^{\Theta}$, then $A \cap B \in D^{\Theta}$ and $A \cup B \in D^{\Theta}$.
3. No other elements belong to D^{Θ}, except those obtained by using rules 1 or 2.

The dual (obtained by switching $\cup$ and $\cap$ in expressions) of D^{Θ} is itself. There are elements in D^{Θ} which are self-dual (dual to themselves), for example α_8 for the case when n = 3 in the following example.

The cardinality of D^{Θ} is majored by 2^{2^n} when the cardinality of Θ equals n, i.e. $|\Theta| = n$. The generation of hyper-power set D^{Θ} is closely related with the famous Dedekind's problem [3, 2] on enumerating the set of isotone Boolean functions. The generation of the hyper-power set is presented in [24]. Since for any given finite set Θ, $|D^{\Theta}| \geq |2^{\Theta}|$ we call D^{Θ} the *hyper-power set* of Θ.

Example of the first hyper-power sets D^{Θ}

• For the degenerate case (n = 0) where $\Theta = \{\}$, one has $D^{\Theta} = \{\alpha_0 \underline{\Delta} \emptyset\}$ and $|D^{\Theta}| = 1$.

• When $\Theta = \{\theta_1\}$, one has $D^{\Theta} = \{\alpha_0 \underline{\Delta} \emptyset, \alpha_1 \underline{\Delta} \theta_1\}$ and $|D^{\Theta}| = 2$.

• When $\Theta = \{\theta_1, \theta_2\}$, one has $D^{\Theta} = \{\alpha_0, \alpha_1, \dots, \alpha_4\}$ and $|D^{\Theta}| = 5$ with $\alpha_0 \underline{\Delta} \emptyset$, $\alpha_1 \underline{\Delta} \theta_1 \cap \theta_2$, $\alpha_2 \underline{\Delta} \theta_1$, $\alpha_3 \underline{\Delta} \theta_2$ and $\alpha_4 \underline{\Delta} \theta_1 \cup \theta_2$.

• When $\Theta = \{\theta_1, \theta_2, \theta_3\}$, one has $D^{\Theta} = \{\alpha_0, \alpha_1, \dots, \alpha_{18}\}$ and $|D^{\Theta}| = 19$ with

$\alpha_0 \underline{\Delta} \emptyset$	$\alpha_9 \underline{\Delta} \theta_1$
$\alpha_1 \underline{\Delta} \theta_1 \cap \theta_2 \cap \theta_3$	$\alpha_{10} \underline{\Delta} \theta_2$
$\alpha_2 \underline{\Delta} \theta_1 \cap \theta_2$	$\alpha_{11} \underline{\Delta} \theta_3$
$\alpha_3 \underline{\Delta} \theta_1 \cap \theta_3$	$\alpha_{12} \underline{\Delta} (\theta_1 \cap \theta_2) \cup \theta_3$
$\alpha_4 \underline{\Delta} \theta_2 \cap \theta_3$	$\alpha_{13} \underline{\Delta} (\theta_1 \cap \theta_3) \cup \theta_2$
$\alpha_5 \underline{\Delta} (\theta_1 \cup \theta_2) \cap \theta_3$	$\alpha_{14} \underline{\Delta} (\theta_2 \cap \theta_3) \cup \theta_1$
$\alpha_6 \underline{\Delta} (\theta_1 \cup \theta_3) \cap \theta_2$	$\alpha_{15} \underline{\Delta} \theta_1 \cup \theta_2$
$\alpha_7 \underline{\Delta} (\theta_2 \cup \theta_3) \cap \theta_1$	$\alpha_{16} \underline{\Delta} \theta_1 \cup \theta_3$
$\alpha_8 \underline{\Delta} (\theta_1 \cap \theta_2) \cup (\theta_1 \cap \theta_3) \cup (\theta_2 \cap \theta_3)$	$\alpha_{17} \underline{\Delta} \theta_2 \cup \theta_3$
	$\alpha_{18} \underline{\Delta} \theta_1 \cup \theta_2 \cup \theta_3$

The cardinality of hyper-power set D^{Θ} for $n \geq 1$ follows the sequence of Dedekind's numbers [19], i.e. 1, 2, 5, 19, 167, 7580, 7828353,... and analytical expression of Dedekind's numbers has been obtained recently by Tombak in [31] (see [24] for details on generation and ordering of D^{Θ}).

2.2. NOTION OF FREE AND HYBRID DSm MODELS

Elements θ_i, $i = 1, \dots, n$ of Θ constitute the finite set of hypotheses/concepts characterizing the fusion problem under consideration. D^{Θ} constitutes what we call the *free DSm model* $\mathcal{M}^f(\Theta)$ and allows to work with fuzzy concepts which depict a continuous and relative intrinsic nature. Such kinds of concepts cannot be precisely refined in an absolute interpretation because of the unapproachable universal truth. However for some particular fusion problems involving discrete concepts, elements θ_i are truly exclusive. In such case, all the exclusivity constraints on θ_i, $i = 1, \dots, n$ have to be included in the previous model to characterize properly the true nature of the fusion problem and to fit it with the reality. By doing this, the hyper-power set D^Θ reduces naturally to the classical power set 2^Θ and this constitutes the most restricted hybrid DSm model, denoted $\mathcal{M}^0(\Theta)$, coinciding with Shafer's model. As an exemple, let's consider the 2D problemwhere $\Theta = \{\theta_1, \theta_2\}$ with $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$ and assume now that θ_1 and θ_2 are truly exclusive (i.e. Shafer's model $\mathcal{M}^0$ holds), then because $\theta_1 \cap \theta_2 \overset{\mathcal{M}^0}{=} \emptyset$, one gets $D^\Theta = \{\emptyset, \theta_1 \cap \theta_2 \overset{\mathcal{M}^0}{=} \emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\} = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\} \equiv 2^\Theta$.

Between the class of fusion problems corresponding to the free DSm model $\mathcal{M}^f(\Theta)$ and the class of fusion problems corresponding to Shafer's model $\mathcal{M}^0(\Theta)$, there exists another wide class of hybrid fusion problems involving in Θ both fuzzy continuous concepts and discrete hypotheses. In such (hybrid) class, some exclusivity constraints and possibly some non-existential constraints (especially when working on dynamic fusion) have to be taken into account. Each hybrid fusion problem of this class will then be characterized by a proper hybrid DSm model $\mathcal{M}(\Theta)$ with $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$ and $\mathcal{M}(\Theta) \neq \mathcal{M}^0(\Theta)$. As simple example of DSm hybrid model, let's consider the 3D case with the frame $\Theta = \{\theta_1, \theta_2, \theta_3\}$ with the model $\mathcal{M} \neq \mathcal{M}^f$ in which we force all possible conjunctions to be empty, but $\theta_1 \cap \theta_2$. This hybrid DSm model is then represented with the following Venn diagram (where boundaries of intersection of θ_1 and θ_2 are not precisely defined if θ_1 and θ_2 represent only fuzzy concepts like *smallness* and *tallness* by example).

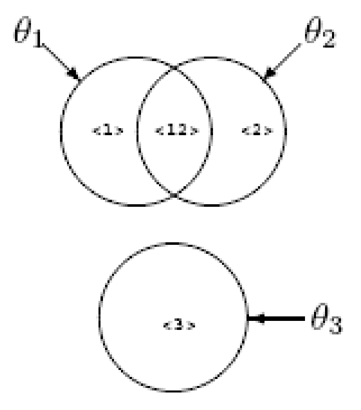

Fig.1 A hybrid DSm model $M^f(\Theta)$

2.3. GENERALIZED BELIEF FUNCTIONS

From a general frame Θ, we define a map $m(.): D^\Theta \rightarrow [0, 1]$ associated to a given body of evidence $\mathcal{B}$ as:

$$m(\emptyset) = 0 \text{ and } \sum_{A \in D^\Theta} m(A) = 1 \qquad (1)$$

The quantity m(A) is called the *generalized basic belief assignment/mass* (gbba) of A. The *generalized belief and plausibility functions* are defined in almost the same manner as within the DST, i.e.

$$\mathrm{Bel}(A) = \sum_{B \subseteq A, B \in D} m(B), \quad \mathrm{Pl}(A) = \sum_{B \cap A \neq \phi, B \in D} m(B) \qquad (2)$$

These definitions are compatible with the definitions of classical belief functions in the DST framework when D^Θ reduces to 2^Θ for fusion problems where Shafer's model $\mathcal{M}^0(\Theta)$ holds.

We still have $\forall A \in D^\Theta$, $\mathrm{Bel}(A) \leq \mathrm{Pl}(A)$. Note that when working with the free DSm model $\mathcal{M}^f(\Theta)$, one has always $\mathrm{Pl}(A) = 1$ $\forall A \neq \emptyset \in D^\Theta$ which is normal.

2.4. THE CLASSIC DSm RULE OF COMBINATION

When the free DSm model $\mathcal{M}^f(\Theta)$ holds for the fusion problem under consideration, the classic DSm rule of combination $m\ \mathcal{M}^f(\Theta) \equiv m(.) \triangleq [m_1 \oplus m_2]$ two independent sources of evidences B_1 and B_2 over the same frame Θ with belief functions $\mathrm{Bel}_1(.)$ and $\mathrm{Bel}_2(.)$ associated with gbba $m_1(.)$ and $m_2(.)$ corresponds to the conjunctive consensus of the sources.

It is given by [24]: $\forall C \in D^\Theta$,

$$m_{M^f(\Theta)}(C) \equiv m(C) = \sum_{A,B \in D^\Theta, A \cap B = C} m_1(A) m_2(B) \qquad (3)$$

Since D^Θ is closed under $\cup$ and $\cap$ set operators, this new rule of combination guarantees that $m(.)$ is a proper generalized belief assignment, i.e. $m(.): D^\Theta \rightarrow [0, 1]$. This rule of combination is commutative and associative and can always be used for the fusion of sources involving fuzzy concepts when free DSm model holds for the problem under consideration. This rule can be directly and easily extended for the combination of $k > 2$ independent sources of evidence [24].

This classic DSm rule of combination looks very expensive in terms of computations and memory size due to the huge number of elements in D^Θ when the cardinality of Θ increases. This remark is however valid only if the cores (the set of focal elements of gbba) $K_1(m_1)$ and $K_2(m_2)$ coincide with D^Θ, i.e. when $m_1(A) > 0$ and $m_2(A) > 0$ for all $A \neq \emptyset \in D^\Theta$. Fortunately, it is important to note here that in most of the practical applications the sizes of $K_1(m_1)$ and $K_2(m_2)$ are much smaller than $|D^\Theta|$ because bodies of evidence generally allocate their basic belief assignments only over a subset of the hyper-power set. This makes things easier for the implementation of the classic DSm rule (3). The DSm rule is actually very easy to implement. It suffices for each focal element of $K_1(m_1)$ to multiply it with the focal elements of $K_2(m_2)$ and then to pool all combinations which are equivalent under the algebra of sets.

While very costly in term on memory storage in the worst case (i.e. when all $m(A) > 0$, $A \in D^\Theta$ or $A \in 2^{\Theta_{ref}}$), the DSm rule however requires much smaller memory storage than for the DST working on the ultimate refinement of $2^{\Theta_{ref}}$ same initial frame Θ as shown in following table:

Table 1 A DSm / DST comparison in memory storage

$\|\Theta\| = n$	$\|D^\Theta\|$	$\|2^{\Theta_{ref}}\| = 2^{2^n - 1}$
2	5	2^3=8
3	19	2^7=128
4	167	2^{15}=32768
5	7580	2^{31}=2147483648

However in most fusion applications only a small subset of elements of D^Θ have a non null basic belief mass because all the commitments are just usually impossible to assess precisely when the dimension of the problem increases. Thus, it is not necessary to generate and keep in memory all elements of D^Θ or $2^{\Theta_{ref}}$ but only those which have a positive belief mass. However there is a real technical challenge on how to manage efficiently all elements of the hyper-power set. This problem is obviously much more difficult when trying to work on the refined frame of discernment $2^{\Theta_{ref}}$ if one prefer to apply Dempster-Shafer theory and use the Dempster's rule of combination. It is important to keep in mind that the ultimate refined frame consisting in exhaustive and exclusive finite set of refined hypotheses is just impossible to justify and to define precisely for all problems dealing with fuzzy and ill-defined continuous concepts. A full discussion and example on refinement can be found in [24].

2.5. THE HYBRID DSm RULE OF COMBINATION

When the free DSm model $\mathcal{M}^f(\Theta)$ does not hold due to the true nature of the fusion problem under consideration which requires to take into account some known integrity constraints, one has to work with a proper hybrid DSm model $\mathcal{M}(\Theta) \neq \mathcal{M}^f(\Theta)$. In such case, the hybrid DSm rule of combination based on the chosen hybrid DSm model $\mathcal{M}(\Theta)$ for $k \geq 2$ independent sources of information is defined for all $A \in D^\Theta$ as [24]:

$$m_{M(\Theta)}(A) \triangleq \phi(A)\,[S_1(A) + S_2(A) + S_3(A)] \qquad (4)$$

where all sets involved in formulas are in the canonical form and $\phi(A)$ is the *characteristic non-emptiness function* of a set A, i.e. $\phi(A) = 1$ if $A \notin \emptyset$ and $\phi(A) = 0$ otherwise, where $\emptyset \triangleq \{\emptyset_M, \emptyset\}$ is the set of all elements of D^Θ which have been forced to be empty through the constraints of the model $\mathcal{M}$ and $\emptyset$ is the classical/universal empty set.

$$S_1(A) \triangleq \sum_{\substack{X_1, X_2, \ldots, X_k \in D^\Theta, \\ X_1 \cap X_2 \cap \ldots \cap X_k = A}} \prod_{i=1}^{k} m_i(X_i) \qquad (5)$$

$$S_2(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in \emptyset, \\ [u=A]\vee[(u\in\emptyset)\wedge(A=I_t)]}} \prod_{i=1}^{k} m_i(X_i) \quad (6)$$

$$S_3(A) \triangleq \sum_{\substack{X_1,X_2,\ldots,X_k \in D^\Theta, \\ X_1\cup X_2\cup\ldots\cup X_k=A, \\ X_1\cap X_2\cap\ldots\cap X_k\in\emptyset}} \prod_{i=1}^{k} m_i(X_i) \quad (7)$$

with $U \triangleq u(X_1) \cup u(X_2) \cup \ldots \cup u(X_k)$ where u (X) is the union of all θ_i that compose X, $I_t \triangleq \theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ is the total ignorance. $S_1(A)$ corresponds to the classic DSm rule for k independent sources based on the free DSm model $\mathcal{M}^f(\Theta)$; $S_2(A)$ represents the mass of all relatively and absolutely empty sets which is transferred to the total or relative ignorances associated with non existential constraints (if any, like in some dynamic problems); $S_3(A)$ transfers the sum of relatively empty sets directly onto the canonical disjunctive form of non-empty sets.

The hybrid DSm rule of combination generalizes the classic DSm rule of combination and is not equivalent to Dempter's rule. It works for any models (the free DSm model, Shafer's model or any other hybrid models) when manipulating *precise* generalized (or eventually classical) basic belief functions. An extension of this rule for the combination of *imprecise* generalized (or eventually classical) basic belief functions is presented in next section.

Note that in DSmT framework it is also possible to deal directly with complements if necessary depending on the problem under consideration and the information provided by the sources of evidence themselves. The first and simplest way is to work on Shafer's model when ultimate refinement is possible. The second way is to deal with partially known frame and introduce directly the complementary hypotheses into the frame itself. By example, if one knows only two hypotheses θ_1, θ_2 and their complements $\overline{\theta_1}$, $\overline{\theta_2}$, then can choose $\Theta = \{\theta_1, \theta_2, \overline{\theta_1}, \overline{\theta_2}\}$. In such case, we don't necessarily assume that $\overline{\theta_1} = \theta_2$ and $\overline{\theta_2} = \theta_1$ because $\overline{\theta_1}$ and $\overline{\theta_2}$ may include other unknown hypotheses we have no information about (case of partial known frame). More generally, in DSmT framework, it is not necessary that the frame is built on pure/simple (possibly vague) hypotheses _i as usually done in all theories managing uncertainty. The frame Θ can also contain directly as elements conjunctions and/or disjunctions (or mixed propositions) and negations/complements of pure hypotheses as well. The DSm rules also work in such non-classic frames because DSmT works on any distributive lattice built from Θ anywhere Θ is defined.

2.6. FUSION OF IMPRECISE BELIEFS

In many fusion problems, it seems very difficult (if not impossible) to have precise sources of evidence generating precise basic belief assignments (especially when belief functions are provided by human experts), and a more flexible plausible and paradoxical theory supporting imprecise information becomes necessary. In the previous sections, we presented the fusion of *precise* uncertain and conflicting/paradoxical generalized basic belief assignments (gbba) in the DSmT framework. We mean here by precise gbba, basic belief functions/masses m(.) defined precisely on the hyper-power set D^Θ, where each mass m(X), where X belongs to D^Θ, is represented by only one real number belonging to [0, 1] such that $\sum_{X\in D^\Theta} m(X) = 1$.

In this section, we present the DSm fusion rule for dealing with *admissible imprecise generalized basic belief assignments* $m^I(.)$ defined as real subunitary intervals of [0, 1], or even more general as real subunitary sets [i.e. sets, not necessarily intervals]. An imprecise belief assignment $m^I(.)$ over D^Θ is said admissible if and only if there exists for every $X \in D^\Theta$ at least one real number m(X) $\in m^I(X)$ such that $\sum_{X\in D^\Theta} m(X) = 1$.

The idea to work with imprecise belief structures represented by real subset intervals of [0, 1] is not new and has been investigated in [11, 4, 5] and references therein. The proposed works available in the literature, upon our knowledge were limited only to sub-unitary interval combination in the framework of Transferable Belief Model (TBM) developed by Smets [29, 30]. We extend the approach of Lamata &Moral and Denoeux based on subunitary interval-valued masses to subunitary set-valued masses; therefore the closed intervals used by Denoeux to denote imprecise masses are generalized to any sets included in [0,1], i.e. in our case these sets can be unions of (closed, open, or half-open/half-closed) intervals and/or scalars all in [0, 1]. Here, the proposed extension is done in the context of the DSmT framework, although it can also apply directly to fusion of imprecise belief structures within TBM as well if the user prefers to adopt TBM rather than DSmT.

Before presenting the general formula for the combination of generalized imprecise belief structures, we remind the following set operators involved in the formula. Several numerical examples are given in [24].

• **Addition of sets**

$S_1 \boxplus S_2 = S_2 \boxplus S_1$, $\{x \mid x = s_1 + s_2,\ s_1 \in S_1,\ s_2 \in S_2\}$ with ($\inf(S_1 \boxplus S_2) = \inf(S_1) + \inf(S_2)$, $\sup(S_1 \boxplus S_2) = \sup(S_1) + \sup(S_2)$)

• **Subtraction of sets**

$S_1 \boxminus S_2$, $\{x \mid x = s_1 - s_2,\ s_1 \in S_1,\ s_2 \in S_2\}$ with ($\inf(S_1 \boxminus S_2) = \inf(S_1) - \sup(S_2)$, $\sup(S_1 \boxminus\_ S_2) = \sup(S_1) - \inf(S_2)$)

• **Multiplication of sets**

$S_1 \boxdot S_2$, $\{x \mid x = s_1 \cdot s_2,\ s_1 \in S_1,\ s_2 \in S_2\}$ with ($\inf(S_1 \boxdot S_2) = \inf(S_1) \cdot \inf(S_2)$, $\sup(S_1 \boxdot S_2) =$ $= \sup(S_1) \cdot \sup(S_2)$)

2.6.1. DSm rule of combination for imprecise beliefs

We present the generalization of the DSm rules to combine any type of imprecise belief assignment which may be represented by the union of several sub-unitary (half-) open intervals, (half-)closed intervals and/or sets of points belonging to [0,1]. Several numerical examples are also given. In the sequel, one uses the notation (a, b) for an open interval, [a, b] for a closed interval, and (a, b] or [a, b) for a half open and half closed interval. From the previous operators on sets, one can generalize the DSm rules (classic and hybrid) from scalars to sets in the following way [24] (chap. 6): $\forall A \neq \emptyset \in D^{\Theta}$,

$$m^I(A) = \sum_{\substack{X_1, X_2, \ldots, X_k \in D^{\Theta},\\ X_1 \cap X_2 \cap \ldots \cap X_k = A}} \prod_{i=1,\ldots,k} m_i^I(X_i) \quad (8)$$

where $\sum$ and $\prod$ represent the summation, and respectively product, of sets. Similarly, one can generalize the hybrid DSm rule from scalars to sets in the following way:

$$m^I_{M(\Theta)}(A) \,\underline{\Delta}\, \emptyset(A) \boxdot [S_1^I(A) \boxplus S_2^I(A) \boxplus S_3^I(A)] \quad (9)$$

where all sets involved in formulas are in the canonical form and $\emptyset(A)$ is the *characteristic non emptiness function* of the set A and $S_1^I(A)$, $S_2^I(A)$ and $S_3^I(A)$ are defined by:

$$S_1^I(A) \,\underline{\Delta} \sum_{\substack{X_1, X_2, \ldots, X_k \in D^{\Theta},\\ X_1 \cap X_2 \cap \ldots \cap X_k = A}} \prod_{i=1,\ldots,k} m_i^I(X_i) \quad (10)$$

$$S_2^I(A) \,\underline{\Delta} \sum_{\substack{X_1, X_2, \ldots, X_k \in \phi,\\ (u=A) \vee (u \in \phi) \wedge (A = I_t) A}} \prod_{i=1,\ldots,k} m_i^I(X_i) \quad (11)$$

$$S_3^I(A) \,\underline{\Delta} \sum_{\substack{X_1, X_2, \ldots, X_k \in D^{\Theta},\\ X_1 \cup X_2 \cup \ldots \cup X_k = A,\\ X_1 \cap X_2 \cap \ldots \cap X_k \in \phi}} \prod_{i=1,\ldots,k} m_i^I(X_i) \quad (12)$$

In the case when all sets are reduced to points (numbers), the set operations become normal operations with numbers; the sets operations are generalizations of numerical operations. When imprecise belief structures reduce to precise belief structure, DSm rules (9) and (10) reduce to their precise version (3) and (4) respectively.

3. PROPORTIONAL CONFLICT REDISTRIBUTION RULE

Instead of applying a direct transfer of partial conflicts onto partial uncertainties as with DSmH, the idea behind the Proportional Conflict Redistribution (PCR) rule [25, 26] is to transfer (total or partial) conflicting masses to non-empty sets involved in the conflicts proportionally with respect to the masses assigned to them by sources as follows:

1. calculation the conjunctive rule of the belief masses of sources;

2. calculation the total or partial conflicting masses;
3. redistribution of the (total or partial) conflictingmasses to the non-empty sets involved in the conflicts proportionally with respect to their masses assigned by the sources.

The way the conflicting mass is redistributed yields actually several versions of PCR rules.

These PCR fusion rules work for any degree of conflict, for any DSm models (Shafer's model, free DSm model or any hybrid DSm model) and both in DST and DSmT frameworks for static or dynamical fusion situations. We present below only the most sophisticated proportional conflict redistribution rule (corresponding to PCR5 in [25, 26] but denoted here just PCR for simplicity) since this PCR rule is what we feel the most efficient PCR fusion rule developed so far6. PCR rule redistributes the partial conflicting mass to the elements involved in the partial conflict, considering the conjunctive normal form of the partial conflict. PCR is what we think the most mathematically exact redistribution of conflicting mass to non-empty sets following the logic of the conjunctive rule.

PCR does a better redistribution of the conflicting mass than Dempster's rule sice PCR goes backwards on the tracks of the conjunctive rule and redistributes the conflicting mass only to the sets involved in the conflict and proportionally to their masses put in the conflict. PCR rule is quasi-associative and preserves the neutral impact of the vacuous belief assignment because in any partial conflict, as well in the total conflict (which is a sum of all partial conflicts), the conjunctive normal form of each partial conflict does not include Θ since Θ is a neutral element for intersection (conflict), therefore Θ gets no mass after the redistribution of the conflicting mass. We have also proved in [26] the continuity property of the PCR result with continuous variations of bba to combine.

The general PCR formula for s $\geq$ 2 sources is given by [26] $m_{PCR}(\phi) = 0$ and $\forall X \in G \setminus \{ ø \}$:

$$m_{PCR}(X) = m_{12\ldots s}(X) + \sum_{\substack{2\le t\le s \\ 1\le r_1,\ldots,r_t \\ \le r_1 \langle r_2 \langle \ldots \\ \langle r_{t-1} \langle (r_t = s)}} \sum_{\substack{X_{j_2,\ldots,j_t} \in G\setminus\{X\}, \\ \{j_2,\ldots,j_t) \in \\ P^{t=1}(\{1,\ldots,n\}) \\ X\cap X_{j_2}\cap\ldots\cap X_{j_s} = \phi, \\ \{i_1,\ldots,i_s\} \in P^s(\{1,\ldots,s\})}} \frac{(\prod_{k_1=1}^{r_1} m_{i_{k_1}}(X)^2)[\prod_{l=2}^{t}(\prod_{k_l=r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]}{(\prod_{k_1=1}^{r_1} m_{i_{k_1}}(X)) + [\sum_{l=2}^{t}(\prod_{k_l=r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]} \tag{13}$$

where all sets involved in formulas are in canonical form (i.e. conjunctive normal form) and where G corresponds to classical power-set 2^{Θ} if Shafer's model is used or G corresponds to a constrained hyper-power set D^{Θ} if any other hybrid DSm model is used instead; i, j, k, r, s and t in (14) are integers. $m_{12\ldots s}(X) \equiv m_{\cap}(X)$ corresponds to the conjunctive consensus on X between s sources and where all denominators are different from zero. If a denominator is zero, that fraction is discarded; the set of all subsets of k elements from {1, 2, . . . , n} (permutations of n elements taken by k) was denoted $P^K(\{1,2,\ldots,n\})$, the order of elements doesn't count. When s = 2 (fusion of only two sources), the previous PCR rule reduces to its simple following fusion formula: $m_{PCR}(\phi) = 0$ and $\forall X \in G \setminus \{ ø \}$,

$$m_{PCR}(X) = m_{12}(X) + \sum_{\substack{Y\in G\setminus\{X\} \\ X\cap Y=\phi}} \left[\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_2(X) + m_1(Y)} \right] \tag{14}$$

3.1. THE GENERALIZED PIGNISTIC TRANSFORMATION (GPT)

3.1.1. The classical pignistic transformation

We follow here the Smets' vision which considers the management of information as a two 2-levels process: credal (for combination of evidences) and pignistic (for decision-making), i.e "*when someone must take a decision, he must then construct a probability* *function derived from the belief function that describes his credal state. This probability function is then used to make decisions*" [28]. One obviousway to build this probability function corresponds to the so-called Classical Pignistic Transformation (CPT) defined in the DST framework (i.e. based on the Shafer's model assumption) as [30]:

$$P\{A\} = \sum_{X \in 2^{\Theta}} \frac{|X \cap A|}{|X|} m(X) \qquad (15)$$

where |A| denotes the number of worlds in the set A (with convention $|\phi|/|\phi| = 1$, to define $P\{\phi\}$). P{A} corresponds to BetP(A) in Smets' notation [30]. Decisions are achieved by computing the expected utilities of the acts using the subjective/pignistic P{.} as the probability function needed to compute expectations. Usually, one uses the maximum of the pignistic probability as decision criterion. The max. of P{.} is often considered as a prudent betting decision criterion between the two other alternatives (max of plausibility or max. of credibility which appears to be respectively too optimistic or too pessimistic). It is easy to show that P{.} is indeed a probability function (see [29]).

3.1.2. Notion of DSm cardinality.

One important notion involved in the definition of the Generalized Pignistic Transformation (GPT) is the *DSm cardinality*. The *DSm cardinality* of any element A of hyper-power set D^{Θ}, denoted $C_M(A)$, corresponds to the number of parts of *A* in the corresponding fuzzy/vague Venn diagram of the problem (model *M*) taking into account the set of integrity constraints (if any), i.e. all the possible intersections due to the nature of the elements θ_i. This *intrinsic cardinality* depends on the model M (free, hybrid or Shafer's model). M is the model that contains A, which depends both on the dimension $n = |\Theta|$ and on the number of non-empty intersections present in its associated Venn diagram (see [24] for details). The DSm cardinality depends on the cardinal of $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$ and on the model of D^{Θ} (i.e., the number of intersections and between what elements of Θ - in a word the structure) at the same time; it is not necessarily that every singleton, say θ_i, has the same DSm cardinal, because each singleton has a different structure; if its structure is the simplest (no intersection of this elements with other elements) then $C_M(\theta_i) = 1$, if the structure is more complicated (many intersections) then $C_M(\theta_i) > 1$; let's consider a singleton θ_i: if it has 1 intersection only then $C_M(\theta_i) = 2$, for 2 intersections only $C_M(\theta_i)$ is 3 or 4 depending on the model M, form intersections it is between m+1 and 2^m depending on the model; the maximum DSm cardinality is 2^{n-1} and occurs for $\theta_1 \cup \theta_2 \cup \ldots \cup \theta_n$ in the free model M^f; similarly for any set from D^{Θ}: the more complicated structure it has, the bigger is the DSm cardinal; thus the DSm cardinality measures the complexity of en element from D^{Θ}, which is a nice characterization in our opinion; we may say that for the singleton θ_i not even $|\Theta|$ counts, but only its structure (= how many other singletons intersect θ_i). Simple illustrative examples are given in Chapter 3 and 7 of [24]. One has $1 \le C_M(A) \le 2^{n-1}$. $C_M(A)$ must not be confused with the classical cardinality |A| of a given set A (i.e. the number of its distinct elements) - that's why a new notation is necessary here. $C_M(A)$ is very easy to compute by programming from the algorithm of generation of D_ given explicated in [24].

4. FUSION OF QUALITATIVE BELIEFS

We introduce here the notion of qualitative belief assignment to model beliefs of human experts expressed in natural language (with linguistic labels). We show how qualitative beliefs can be efficiently combined using an extension of Dezert-Smarandache Theory (DSmT) of plausible and paradoxical quantitative reasoning to qualitative reasoning shortly presented in previous sections. A more detailed presentation can be found in [26]. The derivations are based on a new arithmetic on linguistic labels which allows a direct extension of classical DSm fusion rule or DSm Hybrid rules. An approximate qualitative PCR5 rule is also presented.

4.1. QUALITATIVE OPERATORS

Computing with words (CW) and qualitative information is more vague, less precise than computing with numbers, but it offers the advantage of robustness if done correctly. Here is a general arithmetic we propose for computing with words (i.e. with linguistic labels). Let's consider a finite frame $\Theta = \{\theta_1, ..., \theta_n\}$ of n (exhaustive) elements θ_i, i = 1, 2, . . . , n, with an associated model $M(\Theta)$ on Θ (either Shafer's model $M^0(\Theta)$, free-DSm model $M^f(\Theta)$, or more general any Hybrid-DSm model [24]). A model M(Θ) is defined by the set of integrity constraints on elements of Θ (if any); Shafer's model $M^0(\Theta)$ assumes all elements of Θ truly exclusive, while free-DSm model $M^f(\Theta)$ assumes no exclusivity constraints between elements of the frame Θ. Let's define a finite set of linguistic labels $\bar{L} = \{L_1, L_2, \dots, L_m\}$ where $m \geq 2$ is an integer. $\bar{L}$ is endowed with a total order relationship $\prec$, so that $L_1 \prec L_2 \prec \dots \prec L_m$. To work on a close linguistic set under linguistic addition and multiplication operators, we extends $\bar{L}$ with two extreme values L_0 and L_{m+1} where L_0 corresponds to the minimal qualitative value and L_{m+1} corresponds to the maximal qualitative value, in such a way that $L_0 \prec L_1 \prec L_2 \prec \dots \prec L_m \prec L_{m+1}$ where $\prec$ means inferior to, or less (in quality) than, or smaller (in quality) than, etc. hence a relation of order from a qualitative point of view. But if we make a correspondence between qualitative labels and quantitative values on the scale [0, 1], then $L_{min} = L_0$ would correspond to the numerical value 0, while $L_{max} = L_{m+1}$ would correspond to the numerical value 1, and each L_i would belong to [0, 1], i. e. $L_{min} = L_0 < L_1 < L_2 < \dots < L_m < L_{m+1} = L_{max}$. From now on, we work on extended ordered set L of qualitative values $L = \{L_0, \bar{L}, L_{m+1}\} = \{L_0, L_1, L_2, \dots, L_m, L_{m+1}\}$.

The qualitative addition and multiplication operators are respectively defined in the following way:

• Addition :

$$L_i + L_j = L_{i+j}, \text{ if } i+j < m+1,$$

$$L_i + L_j = L_{m+1}, \text{ if } i+j \geq m+1 \quad (16)$$

• Multiplication :

$$L_i \times L_j = L_{\min\{i,j\}} \quad (17)$$

These two operators are well-defined, commutative, associative, and unitary. Addition of labels is a unitary operation since $L_0 = L_{min}$ is the unitary element, i.e. $L_i + L_0 = L_0 + L_i = L_i + 0 = L_i$ for all $0 \leq i \leq m+1$. Multiplication of labels is also a unitary operation since $L_{m+1} = L_{max}$ is the unitary element, i.e. $L_i \times L_{m+1} = L_{m+1} \times L_i = L_{\min\{i,m+1\}} = L_i$ for $0 \leq i \leq m+1$. L_0 is the unit element for addition, while L_{m+1} is the unit element for multiplication. L is closed under + and ×. The mathematical structure formed by (L, +, ×) is a commutative bisemigroup with different unitary elements for each operation. We recall that a bisemigroup is a set S endowed with two associative binary operations such that S is closed under both operations. If L is not an exhaustive set of qualitative labels, then other labels may exist in between the initial ones, so we can work with labels and numbers - since a refinement of L is possible. When mapping from L to crisp numbers or intervals, $L_0 = 0$ and $L_{m+1} = 1$, while $0 < L_i < 1$, for all i, as crisp numbers, or L_i included in [0, 1] as intervals/subsets. For example, L_1, L_2, L_3 and L_4 may represent the following qualitative values: $L_1 \underline{\Delta}$ very poor, $L_2 \underline{\Delta}$ poor, $L_3 \underline{\Delta}$ good and $L_4 \underline{\Delta}$ very good where $\underline{\Delta}$ symbol means "by definition". We think it is better to define the multiplication × of $L_i \times L_j$ by $L_{\min\{i,j\}}$ because multiplying two numbers a and b in [0, 1] one gets a result which is less than each of them, the product is not bigger than both of them as Bolanos et al. did in [1] by approximating $L_i \times L_j = L_{i+j} >$ $> \max\{L_i, L_j\}$.

While for the addition it is the opposite: adding two numbers in the interval [0, 1] the sum should be bigger than both of them, not smaller as in [1] case where $L_i + L_j = \min\{L_i, L_j\} < \max\{L_i, L_j\}$.

4.2. QUALITATIVE BELIEF ASSIGNMENT

A qualitative belief assignment (qba), and we call it qualitative belief mass or q-mass for short, is a mapping function qm(.): $G^\Theta \mapsto L$ where G^Θ corresponds the space of propositions generated with $\cap$ and $\cup$ operators and elements of Θ taking into account the integrity constraints of the model. For example if Shafer's model is chosen for Θ, then G^Θ is nothing but the classical power set 2^Θ [18], whereas if free DSm model is adopted G^Θ will correspond to Dedekind's lattice (hyper-power set) D^Θ [24].

Note that in this qualitative framework, there is no way to define normalized qm(.), but qualitative quasi-normalization is still possible as seen further.

Using the qualitative operations defined previously we can easily extend the combination rules from quantitative to qualitative. In the sequel we will consider $s \geq 2$ qualitative belief assignments $qm_1(.),\ldots,$ $qm_s(.)$ defined over the same space G^Θ and provided by s independent sources $S_1,\ldots,S_s$ of evidence.

Important note:

The addition and multiplication operators used in all qualitative fusion formulas in next sections correspond to qualitative addition and qualitative multiplication operators defined in (18) and (19) and must not be confused with classical addition and multiplication operators for numbers.

4.3. QUALITATIVE CONJUNCTIVE RULE

The qualitative Conjunctive Rule (qCR) of $s \geq 2$ sources is defined similarly to the quantitative conjunctive consensus rule, i.e.:

$$qm_{qCR}(X) = \sum_{\substack{X_1,\ldots,X_s \in G^\Theta,\\ X_1\cap\ldots\cap X_s = X}} \prod_{i=1}^{s} qm_i(X_i) \quad (18)$$

The total qualitative conflicting mass is given by:

$$K_{1\ldots s} = \sum_{\substack{X_1,\ldots,X_s \in G^\Theta,\\ X_1\cap\ldots\cap X_s = \phi}} \prod_{i=1}^{s} qm_i(X_i)$$

4.4. QUALITATIVE DSm CLASSIC RULE

The qualitative DSm Classic rule (q-DSmC) for $s \geq 2$ is defined similarly to DSm Classic fusion rule (DSmC) as follows: $qm_{qDSmC}(\phi) = L_0$ and for all $X \in D^\Theta \setminus \{\phi\}$,

$$qm_{qDSmC}(X) = \sum_{\substack{X_1,\ldots,X_s \in D^\Theta,\\ X_1\cap\ldots\cap X_s = X}} \prod_{i=1}^{s} qm_i(X_i) \quad (19)$$

4.5. QUALITATIVE DSm HYBRID RULE

The qualitative DSm Hybrid rule (q-DSmH) is defined similarly to quantitative DSm hybrid rule [24] as follows:

$$qm_{qDSmH}(\phi) = L_0 \quad (20)$$

and for all $X \in G^\Theta \setminus \{\phi\}$,

$$qm_{qDSmH}(X) \triangleq \phi(X)\,[qS_1(X) + qS_2(X) + qS_3(X)] \quad (21)$$

where all sets involved in formulas are in the canonical form and $\phi(X)$ is the characteristic non-emptiness function of a set X, i.e.: $\phi(X) = L_{m+1}$ if $X \notin \phi$ and $\phi(X) = L_0$ otherwise, where $\phi \triangleq \{\phi_M, \phi\}$. ϕ_M is the set of all elements of D^Θ which have been forced to be empty through the constraints of the model M and ϕ is the classical/universal empty set.

$qS_1(X) \equiv qm_{qDSmC}(X)$, $qS_2(X)$, $qS_3(X)$ are defined by:

$$qS_1(X) \triangleq \sum_{\substack{X_1,\ldots,X_s \in D^\Theta \\ X_1 \cap \ldots \cap X_s = X}} \prod_{i=1}^{s} qm_i(X_i) \quad (22)$$

$$qS_2(X) \triangleq \sum_{\substack{X_1,\ldots,X_s \in \phi, \\ (u=X) \vee [(u \in \phi) \wedge (X = I_t)]}} \prod_{i=1}^{s} qm_i(X_i) \quad (23)$$

$$qS_3(X) \triangleq \sum_{\substack{X_1,\ldots,X_s \in D^\Theta, \\ X_1 \cup \ldots \cup X_s = X \\ X_1 \cap \ldots \cap X_s \in \phi}} \prod_{i=1}^{s} qm_i(X_i) \quad (24)$$

with $u \triangleq u(X_1) \cup \ldots \cup u(X_s)$ where u(X) is the union of all θ_i that compose X, $I_t \triangleq \theta_i \cup \ldots \cup \theta_n$ is the total ignorance. $qS_1(X)$ is nothing but the qDSmC rule for s independent sources based on $M^f(\Theta)$; $qS_2(X)$ is the qualitative mass of all relatively and absolutely empty sets which is transferred to the total or relative ignorances associated with non existential constraints (if any, like in some dynamic problems); $qS_3(X)$ transfers the sum of relatively empty sets directly onto the canonical disjunctive form of non-empty sets. qDSmH generalizes qDSmC works for any models (free DSm model, Shafer's model or any hybrid models) when manipulating qualitative belief assignments.

4.6. QUALITATIVE PCR5 RULE (Q - PCR5)

In classical (i.e. quantitative) DSmT framework, the Proportional Conflict Redistribution rule no. 5 (PCR5) defined in [26] has been proven to provide very good and coherent results for combining (quantitative) belief masses, see [25, 7].

When dealing with qualitative beliefs and using Dempster-Shafer Theory (DST), we unfortunately cannot normalize, since it is not possible to divide linguistic labels by linguistic labels.

Previous authors have used the un-normalized Dempster's rule, which actually is equivalent to the Conjunctive Rule in Shafer's model and respectively to DSm conjunctive rule in hybrid and free DSm models. Following the idea of (quantitative) PCR5 fusion rule, we can however use a rough approximation for a qualitative version of PCR5 (denoted qPCR5) as it will be presented in next example, but we did not succeed so far to get a general formula for qualitative PCR5 fusion rule (q-PCR5) because the division of labels could not be defined.

5. CONCLUSION

A general presentation of foundation of DSmT has been proposed in this introduction which proposes new quantitative rules of combination for uncertain, imprecise and highly conflicting sources of information. Several applications of DSmT have been proposed recently in the literature and show the efficiency of this new approach over classical rules, mainly those based on the Demspter's rule in the DST framework. Recent PCR rules of combination (typically PCR no 5) have also been developed which offer a more precise transfer of partial conflicts than classical rules. DSmT rules have been also extented for the fusion of qualitative beliefs expressed in terms of linguistic labels for dealing directly with natural language and human reports. Matlab source code for the implementation of DSm rules and also new belief conditioning rules (not presented herein) have been recently developed and can be found in the forthcoming second DSmT book [26].

REFERENCES

1. Bolanos, J., De Campos, L.M., Moral, S., *Propagation of linguistic labels in causal networks*, Proc. of 2nd IEEE Int. Conf. on Fuzzy Systems, Vol. 2, pp. 863–870, 28 March–1 April 1993;
2. Comtet, L., *Sperner Systems*, sec.7.2 in Advanced Combinatorics: The Art of Finite and Infinite Expansions, D. Reidel Publ. Co., pp.271-273, 1974;
3. Dedekind, R., *Ber Zerlegungen von Zahlen durch ihre grssten gemeinsammen Teiler*, In Gesammelte Werke, Bd. 1,1897;

4. Denoeux, T, *Reasoning with imprecise belief structures*, Technical Report Heudiasys 97/44, available at http://www.hds.utc.fr/˜tdenoeux/;
5. Denoeux, T, *Reasoning with imprecise belief structures*, International Journal of Approximate Reasoning, 20, pp. 79-111, 1999;
6. Dezert, J., Smarandache, F., *DSmT: A New Paradigm Shift for Information Fusion*, in Proceedings of Cogis '06 Conference, Paris, March 2006;
7. Dezert, J., Tchamova, A., Smarandache, F., Konstantinova, P., *Target Type Tracking with PCR5 and Dempster's rules: A Comparative Analysis*, in Proceedings of Fusion 2006 International conference on Information Fusion, Fusion 2006, Firenze, Italy, July 10-13, 2006;
8. Dubois, D., Prade, H., *On the unicity of Dempster rule of combination*, International Journal of Intelligent Systems, Vol. 1, pp 133-142, 1986;
9. Dubois, D., Prade, H., *Representation and combination of uncertainty with belief functions and possibility measures*, Computational Intelligence, 4, pp. 244-264, 1988;
10. Inagaki, T., *Interdependence between safety-control policy and multiple-sensor schemes via Dempster-Shafer theory*, IEEE Trans. on reliability, Vol. 40, no. 2, pp. 182-188, 1991;
11. Lamata, M., Moral, S., *Calculus with linguistic probabilities and beliefs*, In R. R. Yager,M. Fedrizzi, and J. Kacprzyk, editors, Advances in Dempster-Shafer Theory of Evidence, pp. 133-152, Wiley;
12. Lefevre, E., Colot, O., Vannoorenberghe, P., *Belief functions combination and conflictmanagement*, Information Fusion Journal, Elsevier Publisher, Vol. 3, No. 2, pp. 149-162, 2002;
13. Lefevre, E., Colot, O., Vannoorenberghe, P., *Reply to the Comments of R. Haenni on the paper "Belief functions combination and conflict management"*, Information Fusion Journal, Elsevier Publisher, Vol. 4, pp. 63-65, 2003;
14. Murphy, C.K., *Combining belief functions when evidence conflicts*, Decision Support Systems, Elsevier Publisher, Vol. 29, pp. 1-9, 2000;
15. Pearl, J., *Probabilistic reasoning in Intelligent Systems: Networks of Plausible Inference*, Morgan Kaufmann Publishers, San Mateo, CA, 1988;
16. Robinson, A., *Non-Standard Analysis*, North-Holland Publ. Co., 1966;
17. Sentz, K., Ferson, S., *Combination of evidence in Dempster-Shafer Theory*, SANDIA Tech. Report, SAND2002-0835, 96 pages, April 2002;
18. Shafer, G., *A Mathematical Theory of Evidence*, Princeton Univ. Press, Princeton, NJ, 1976;
19. Sloane, N.J.A., *The On-line Encyclopedia of Integer Sequences 2003*, (Sequence No. A014466),http://www.research.att.com/˜njas/sequences/;
20. Smarandache, F., *A Unifying Field in Logics: Neutrosophic Logic. Neutrosophy, Neutrosophic Set, Probability, and Statistics*, (2nd Ed.), Amer. Research Press, Rehoboth, 2000;
21. Smarandache, F., *A Unifying Field in Logics: Neutrosophic Logic*, Multiple-valued logic, An international journal, Vol. 8, No. 3, pp. 385-438, 2002;
22. Smarandache, F., *Neutrosophy: A new branch of philosophy*, Multiple-valued logic, An international journal, Vol. 8, No. 3, pp. 297-384, 2002;
23. Smarandache, F. (Editor), *Proceedings of the First International Conference on Neutrosophics*, Univ. of NewMexico, Gallup Campus, NM, USA, 1-3 Dec. 2001, Xiquan, Phoenix, 2002;
24. Smarandache, F., Dezert J. (Editors), *Applications and Advances of DSmT for Information Fusion*, Am. Res. Press, Rehoboth, 2004, http://www.gallup.unm.edu/˜smarandache/DSmT-book1.pdf;
25. Smarandache, F., Dezert, J., *Information Fusion Based on New Proportional Conflict Redistribution Rules*, Proceedings of Fusion 2005 Conf., Philadelphia, July 26-29, 2005;
26. Smarandache, F., Dezert, J. (Editors), *Applications and Advances of DSmT for Information Fusion*, Vol. 2, American Research Press, Rehoboth, August 2006;

27. Smets, Ph., *Combining non distinct evidence*, Proc. North American Fuzzy Information Processing (NAFIP 1986), New Orleans, LA, 1986;
28. Smets, Ph., Mamdani, E.H., Dubois, D., Prade, H. (Editors), *Non-Standard Logics for Automated Reasoning*, Academic Press, 1988;
29. Smets, Ph., Kennes, R., *The transferable belief model*, Artif. Intel., 66(2), pp. 191-234, 1994;
30. Smets, Ph., *Data Fusion in the Transferable Belief Model*, Proceedings of the 3rd International Conference on Information Fusion, Fusion 2000, Paris, July 10-13, 2000, pp PS21-PS33;
31. Tombak, M., Isotamm, A., Tamme, T., *On logical method for counting Dedekind numbers*, Lect. Notes on Comp.Sci., 2138, p. 424-427, Springer-Verlag, 2001. www.cs.ut.ee/people/m_tombak/publ.html;
32. Voorbraak, F., *On the justification of Dempster's rule of combination*, Artificial Intelligence, 48, pp. 171-197, 1991. (see http://turing.wins.uva.nl/~fransv/#pub);
33. Yager, R.R., *Hedging in the combination of evidence*, Journal of Information and Optimization Science, Vol. 4, No. 1, pp. 73-81, 1983;
34. Yager, R.R., *On the relationships of methods of aggregation of evidence in expert systems*, Cybernetics and Systems, Vol. 16, pp. 1-21, 1985;
35. Yager, R.R., *On the Dempster-Shafer framework and new combination rules*, Information Sciences, Vol. 41;
36. Zadeh, L., *Fuzzy sets*, Inform and Control 8, pp. 338-353, 1965;
37. Zadeh, L., *Fuzzy Logic and Approximate Reasoning*, Synthese, 30, 407-428, 1975;
38. Zadeh, L., *On the validity of Dempster's rule of combination*,Memo M 79/24, Univ. of California, Berkeley, 1979;
39. Zadeh, L., *Review of Mathematical theory of evidence, by Glenn Shafer*, AIMagazine, Vol. 5, No. 3, pp. 81-83, 1984;
40. Zadeh, L., *A simple view of the Dempster-Shafer theory of evidence and its implications for the rule of combination*, Berkeley Cognitive Science Report No. 33, University of California, Berkeley, CA, 1985.Zadeh L., *A simple view of the Dempster-Shafer theory of evidence and its implication for the rule of combination*, AI Magazine 7, No.2, pp. 85-90, 1986;
41. Zhang, L., *Representation, independence, and combination of evidence in the Dempster-Shafer Theory*, Advances in the Dempster-Shafer Theory of Evidence, R.R. Yager, J. Kacprzyk and M. Fedrizzi, Eds., John Wiley and Sons, Inc., New York, pp. 51-69, 1994.