THE RAYLEIGH'S FAMILY OF DISTRIBUTIONS

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Abstract: *Several general mathematical properties of the Rayleigh's family of distributions are examined in a consistent manner by using the power series distributions (PSD) class [1]. A new cumulative distribution function and probability density are obtained for the continuous type random variables which represent the maximum or the minimum in a sequence of independent, identically Rayleigh distributed random variables, in a random number by means of a power series distribution. An asymptotic result characterized by the Poisson Limit Theorem is formulated and analysed.*

Keywords: *power series distributions, Rayleigh distribution, distribution of the maximum and minimum, Poisson Limit Theorem*

1. INTRODUCTION

In the paper [2] sets out to introduce and analyse the properties of the maximum and minimum distributions for a sample of power series distribution. This can serve as a mathematical model to describe the probabilistic behaviour of the signals used on a large scale in the field of radiolocation. In this paper, the distribution is presented as being the distribution of the maximum or minimum value from a sample of random volume Z from a Rayleigh distributed statistical population, where Z is a random value from the power series distribution class.

2. MIN RAYLEIGH AND MAX RAYLEIGH POWER SERIES DISTRIBUTIONS

It is know that a random variable admits a Rayleigh distribution with the parameter σ, and we note $X : Rayleigh(\sigma), \sigma > 0$, if the cumulative distribution function (cdf) is $F_{Ray}(x) = 1 - e^{-\frac{x^2}{2\sigma^2}}, x \geq 0$, while the corresponding probability density function (pdf) $f_{Ray}(x) = \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}, x \geq 0$.

We consider the random variables $U_{Ray} = \max\{X_1, X_2, ..., X_Z\}$ and $V_{Ray} = \min\{X_1, X_2, ..., X_Z\}$, where $(X_i)_{i\geq 1}$ are independent and identically distributed random variables, $X_i : Rayleigh\ (\sigma), \sigma > 0$ and $Z \in PSD$, that is $P(Z = z) = \frac{a_z \theta^z}{A(\theta)}, z = 1,2,...,$ where $a_1, a_2, ...$ is a sequence of real, non-negative numbers. $\tau > 0$ the radius of convergence of the power series $A(\Theta) = \sum_{z\geq 1} a_z \Theta^z$, $\forall\ \Theta \in (0, \tau)$ and Θ the real parameter of the distribution.

We point out that the random variables $(X_i)_{i\geq 1}$ are independent of the random variable Z, the latter's distribution being part of the power series distributions class [1].

In accordance with the working methods in the paper [2,4], it can be stated that the random variables U_{Ray} follow the *Max Rayleigh power series distributions* of parameters σ and Θ (we note: U_{Ray} : $MaxRayleighPS\ (\sigma,\Theta)$) and V_{Ray} follow the *Min Rayleigh power series distributions* of parameters σ and Θ (we note: V_{Ray} : $MinRayleighPS\ (\sigma,\Theta)$) if the cumulative distribution functions (cdf) are characterized by the relation:

$$U_{Ray}(x)=\frac{A\left(\Theta F_{Ray}(x)\right)}{A(\Theta)}=\frac{A\left[\Theta\left(1-e^{-\frac{x^2}{2\sigma^2}}\right)\right]}{A(\Theta)},\ x\geq 0 \tag{1}$$

and

$$V_{Ray}(x)=\frac{A(\Theta)-A\left[\Theta\left(1-F_{Ray}(x)\right)\right]}{A(\Theta)}=\frac{A(\Theta)-A\left[\Theta e^{-\frac{x^2}{2\sigma^2}}\right]}{A(\Theta)},\ x\geq 0. \tag{2}$$

The probability densities functions (pdf) are characterized by the relation:

$$u_{Ray}(x)=\frac{\Theta\cdot f_{Ray}(x)\cdot\frac{d}{dx}\left[A\left(\Theta F_{Ray}(x)\right)\right]}{A(\Theta)}=\frac{\Theta x e^{-\frac{x^2}{2\sigma^2}}\frac{d}{dx}\left\{A\left[\Theta\left(1-e^{-\frac{x^2}{2\sigma^2}}\right)\right]\right\}}{\sigma^2 A(\Theta)},\ x\geq 0\ , \tag{3}$$

and,

$$v_{Ray}(x)=\frac{\Theta\cdot f_{Ray}(x)\cdot\frac{d}{dx}\left\{A\left[\Theta\left(1-F_{Ray}(x)\right)\right]\right\}}{A(\Theta)}=\frac{\Theta\cdot x\cdot e^{-\frac{x^2}{2\sigma^2}}\cdot\frac{d}{dx}\left[A\left(\Theta e^{-\frac{x^2}{2\sigma^2}}\right)\right]}{\sigma^2 A(\Theta)},\ x\geq 0. \tag{4}$$

Proposition 2.1. If $(X_i)_{i\geq 1}$ is a sequence of independent random variables, Rayleigh distributed with parameters $\sigma>0$**,** while $U_{Ray}=\max\left\{X_1,X_2,...,X_Z\right\}$ where $Z\in PSD$ with $P(Z=z)=\frac{a_z\theta^z}{A(\theta)},\ z=1,2,...,$ $A(\Theta)=\sum_{z\geq 1}a_z\Theta^z$, $\forall\ \Theta\in(0,\tau)$, then:

$$\lim_{\Theta\to 0^+}U_{Ray}(x)=\left[1-e^{-\frac{x^2}{2\sigma^2}}\right]^k,\ x\geq 0,\quad \text{where } k=\min\left\{k\in N^*,\ a_k>0\right\}.$$

Proposition 2.2. If $(X_i)_{i\geq 1}$ is a sequence of independent random variables, Rayleigh distributed with parameters $\sigma>0$**,** while $V_{Ray}=\max\left\{X_1,X_2,...,X_Z\right\}$ where $Z\in PSD$ with $P(Z=z)=\frac{a_z\theta^z}{A(\theta)},\ z=1,2,...,$ $A(\Theta)=\sum_{z\geq 1}a_z\Theta^z$, $\forall\ \Theta\in(0,\tau)$, then:

$$\lim_{\Theta\to 0^+}V_{Ray}(x)=1-e^{-\frac{x^2 l}{2\sigma^2}},\ x\geq 0,\ \text{where } l=\min\left\{l\in N^*,\ a_l>0\right\}.$$

Corollary 2.1. The r^{th} moments, $r \in N$, $r \geq 1$ of the random variables U_{Ray} : $MaxRayleighPS(\sigma, \Theta)$ and V_{Ray} : $MinRayleighPS(\sigma, \Theta)$ are given by:

$$EU_{Ray}^r = \sum_{z \geq 1} \frac{a_z \Theta^z}{A(\Theta)} E\left\{\max\left[X_1, X_2, ..., X_z\right]\right\}^r$$

and

$$EV_{Ray}^r = \sum_{z \geq 1} \frac{a_z \Theta^z}{A(\Theta)} E\left\{\min\left[X_1, X_2, ..., X_z\right]\right\}^r,$$

where the pdfs of the random variables $\max\left[X_1, X_2, ..., X_z\right]$ and $\min\left[X_1, X_2, ..., X_z\right]$ are $f_{\max[X_1, X_2, ..., X_z]}(x) = z\, f_{Ray}(x)\left[F_{Ray}(x)\right]^{z-1}$ and $f_{\min[X_1, X_2, ..., X_z]}(x) = z\, f_{Ray}(x)\left[1 - F_{Ray}(x)\right]^{z-1}$.

3. SPECIAL CASES

3.1. The Max Rayleigh Binomial and Min Rayleigh Binomial distributions. The Max Rayleigh Binomial (MaxRayB) and Min Rayleigh Poisson (MinRayP) distributions are defined by the distribution functions presented in a general framework in [2], where $A(\Theta) = \left(\Theta + 1\right)^n - 1$, with $\Theta = \frac{p}{1-p}$, $p \in (0,1)$:

$$U_{RayB}(x) = \frac{A\left(\Theta F_{Ray}(x)\right)}{A(\Theta)} = \frac{\left[\Theta\left(1 - e^{-\frac{x^2}{2\sigma^2}}\right) + 1\right]^n - 1}{(\Theta+1)^n - 1}$$
$$= \frac{\left(1 - p \cdot e^{-\frac{x^2}{2\sigma^2}}\right)^n - (1-p)^n}{1 - (1-p)^n}, \; x \geq 0 \tag{5}$$

and

$$V_{RayB}(x) = 1 - \frac{A\left[\Theta\left(1 - F_{Ray}(x)\right)\right]}{A(\Theta)} = \frac{\left(\Theta+1\right)^n - \left(\Theta e^{-\frac{x^2}{2\sigma^2}} + 1\right)^n}{(\Theta+1)^n - 1}$$
$$= \frac{1 - \left(1 - p + p \cdot e^{-\frac{x^2}{2\sigma^2}}\right)^n}{1 - (1-p)^n}, \; x \geq 0. \tag{6}$$

respectively.

3.2. The Max Rayleigh Poisson and Min Rayleigh Poisson distributions. The Max Rayleigh Poisson (MaxRayP) and Min Rayleigh Poisson (MinRayP) distributions are characterized by the cumulative distributions functions defined by the relations (1) and (2), where $A(\Theta^*) = e^{\Theta^*} - 1$, with $\Theta^* = \lambda$, $\lambda > 0$:

$$U_{RayP}(x) = \frac{A\left(\Theta^* F_{Ray}(x)\right)}{A(\Theta^*)} = \frac{e^{\Theta^* F_{Ray}(x)} - 1}{e^{\Theta^*} - 1}$$
$$= \frac{e^{-\lambda e^{-\frac{x^2}{2\sigma^2}}} - e^{-\lambda}}{1 - e^{-\lambda}}, \; x \geq 0 \tag{7}$$

and

$$V_{RayP}(x)=1-\frac{A\left[\Theta^*\left(1-F_{Ray}(x)\right)\right]}{A(\Theta^*)}=\frac{1-e^{-\lambda\left(1-e^{-\frac{x^2}{2\sigma^2}}\right)}}{1-e^{-\lambda}},\ x\geq 0. \tag{8}$$

3.3. On the Poisson limit theorem. The following theorems show that the MaxRayP and MinRayP distributions approximate the MaxRayB and MinRayB distributions depending on certain conditions.

Theorem 3.1. (Poisson limit theorem). *The MaxRayP and MinRayP distributions can be obtained as the limit of the MaxRayB, respectively MinRayB distributions with distribution functions given by (5) and (6) if* $n\cdot\Theta\to\lambda$ when $n\to\infty$ and $\Theta\to 0^+$.

In other words,

(i) $\lim\limits_{\substack{n\to\infty\\ p\to 0^+}} V_{RayB}(x)=V_{RayP}(x),\ \forall x\geq 0$, *where* $V_{RayB}(x)$ *and* $V_{RayP}(x),\ x\geq 0$ *are the distribution functions of the random variables* V_{RayB} : $MinRayleighB(\sigma,n,p)$ *and* V_{RayP} : $MinRayleighPoi(\sigma,\lambda)$ *defined by (6) and (8);*

(ii) $\lim\limits_{\substack{n\to\infty\\ p\to 0^+}} U_{RayB}(x)=U_{RayP}(x),\ \forall x\geq 0$, *where* $U_{RayB}(x)$ *and* $U_{RayP}(x),\ x\geq 0$ *are t distribution functions of the random variables* U_{RayB} : $MaxRayleighB(\sigma,n,p)$ *and* U_{RayP} : $MaxRayleighPoi(\sigma,\lambda)$ *defined by (5) and (7).*

Proof. We examine the convergence in terms of the maximum distributions $U_{RayB}(x)$ and $U_{RayP}(x)$, $x\geq 0$.

It is evident that:

$$\lim_{\substack{n\to\infty\\ p\to 0^+}}(1-p)^n=\lim_{\substack{n\to\infty\\ p\to 0^+}}\left[(1-p)^{-1/p}\right]^{-np}=e^{-\lambda};$$

$$\lim_{\substack{n\to\infty\\ p\to 0^+}}(1-p\cdot e^{-x^2/2\sigma^2})^n=\lim_{\substack{n\to\infty\\ p\to 0^+}}\left[\left(1-p\cdot e^{-x^2/2\sigma^2}\right)^{-1/p\cdot e^{-x^2/2\sigma^2}}\right]^{-np\cdot e^{-x^2/2\sigma^2}}$$

$$=e^{-\lim\limits_{\substack{n\to\infty\\ p\to 0^+}} np\cdot e^{-x^2/2\sigma^2}}=e^{-\lambda\cdot e^{-x^2/2\sigma^2}}.$$

$$\lim_{\substack{n\to\infty\\ p\to 0^+}}U_{RayB}(x)=\lim_{\substack{n\to\infty\\ p\to 0^+}}\frac{\left(1-pe^{-x^2/2\sigma^2}\right)^n-(1-p)^n}{1-(1-p)^n}=\frac{e^{-\lambda\cdot e^{-x^2/2\sigma^2}}-e^{-\lambda}}{1-e^{-\lambda}}=U_{RayP}(x),\ x\geq 0.$$

Fig. 1 and 2 show the behaviour of the pdfs of $MinRayleighB(\sigma,n,p)$, $MinRayleighPoi\ (\sigma,\lambda)$, $MaxRayleighB(\sigma,n,p)$ and $MaxRayleighPoi(\sigma,\lambda)$ for some values of the parameters: $n=40$, $p=\frac{1}{10}$, $\lambda=4$, $\sigma=5$.

Fig. 3 and 4 show the behaviour of the pdfs of $MinRayleighB(\sigma,n,p)$, $MinRayleighPoi\ (\sigma,\lambda)$, $MaxRayleighB(\sigma,n,p)$ and $MaxRayleighPoi(\sigma,\lambda)$ for some values of the parameters.

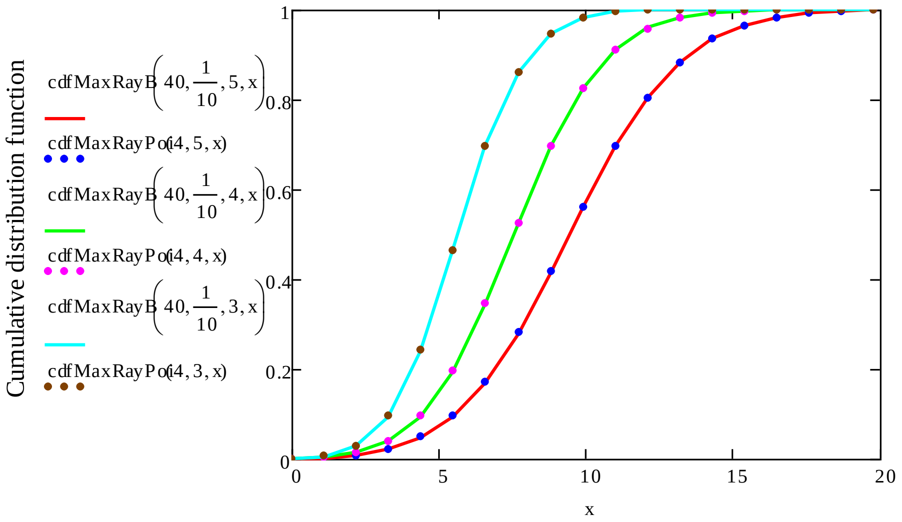

Fig. 1: Pdfs for the Max-Rayleigh-Binomial and Max-Rayleigh-Poisson distributions – graphical illustration of the Poisson Limit Theorem

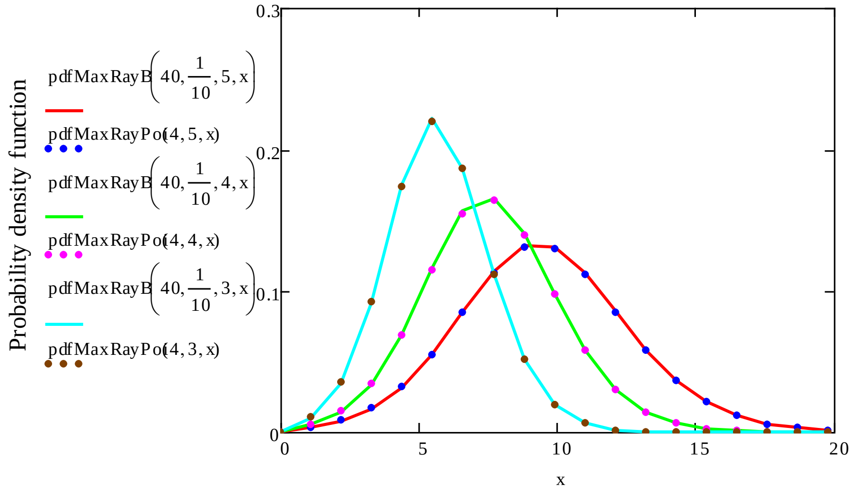

Fig. 2: Pdfs for the Max-Rayleigh-Binomial and Max-Rayleigh-Poisson distributions – graphical illustration of the Poisson Limit Theorem

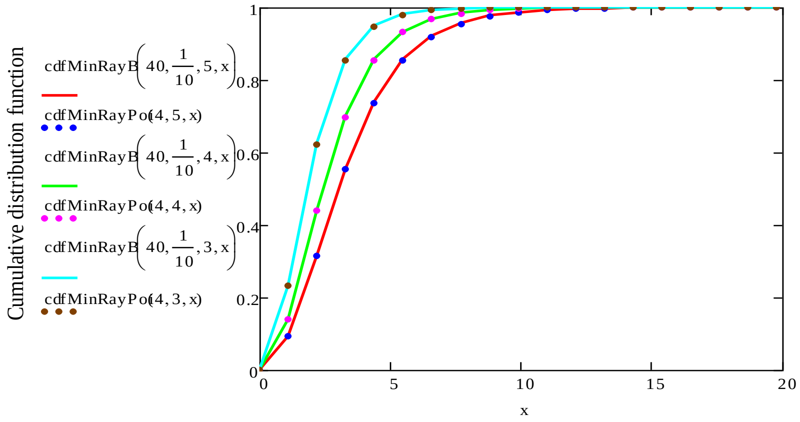

Fig. 3: Pdfs for the Min-Rayleigh-Binomial and Min-Rayleigh-Poisson distributions – graphical illustration of the Poisson Limit Theorem

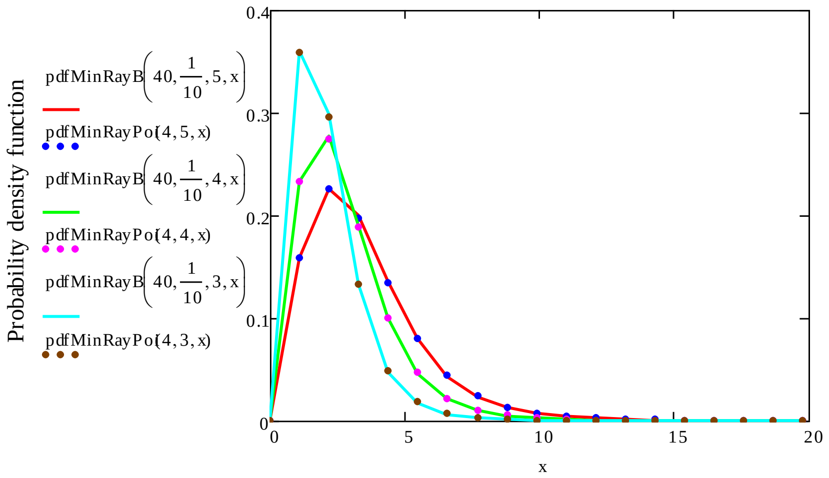

Fig. 4: Pdfs for the Min-Rayleigh-Binomial and Min-Rayleigh-Poisson distributions – graphical illustration of the Poisson Limit Theorem

CONCLUSIONS

The results formulated and examined in this paper are in connection with the study of the random variable distribution, which can be expressed as being the maximum or minimum of a sequence of independent random variables identically distributed in a random number. In practice, this translates as the emission and reception of some signals is a random number, signals whose amplitude is a random variable characterized by the Rayleigh distribution [3]. The signals that best record either a maximum or minimum amplitude are of special interest to us.

It has, thus, been presented in a consistent manner how to determine the maximum and minimum distribution of independent and identically distributed random variables, which form a random sequence.

The Poisson Limit Theorem has been formulated when the random variable number in a sequence has a zero truncated binomial distribution and the limit distribution of the minimum and maximum is of Poisson type.

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