ON MIXING CONTINUOUS DISTRIBUTIONS WITH DISCRETE DISTRIBUTIONS USED IN RELIABILITY

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Abstract: *In this paper, new distributions with applications in the reliability of multi-component systems (and not only!) are obtained using the composition of two probability distributions. We consider the composition between: a) truncated binary distribution (Bin(n,p)) with Lindley distribution (Lindley(θ)), b) Kemp distribution (Kemp(α)) with exponential distribution, (Exp(λ)) and c) truncated Zipf distribution (Zipf(α,n)) with exponential distribution (Exp(λ)). Algorithms for numerical simulation of these probability distributions and some comparisons between their performances are presented.*

Keywords: *truncated binomial discrete distribution, Kemp distribution, Zipf truncated distribution, inverse method, composition method, lifetime variables*

1. INTRODUCTION

Lindley (1958, 1965) [6], [7] introduced a new probability distribution that eventually triggered the interest of researchers. Known as the Lindley distribution, this new distribution was used in modelling system reliability [2]. Many researchers, including Faton Merovci [3] and a group of Romanian researchers coordinated by Professor Vasile Preda [8], introduced some generalizations of this distribution by gaining new divisions that proved appropriate in modelling practical situations. It is well known that the exponential distribution has wide applications in reliability. These distributions will be composed of truncated binomial discrete distribution, Kemp and truncated Zipf [4].

The Zipf (α, n) truncated distribution has applications in situations such as: in a statistical population a small number of individuals have a high frequency property, a large number of individuals occasionally have that property and a large number of individuals rarely have that property [12].

In this paper we consider that the lifetimes of the components of a system with n parallel-connected components are random and identically distributed variables (iid) with either Lindley distribution or exponential distribution. The number of independent components is also considered to be a random variable whose distribution is, in turn, Bin (n, p), Kemp (θ), respectively Zipf (α, n) [5], [11].

2. COMPOSITION OF PROBABILITY DISTRIBUTIONS

2.1. The *Bin-Lindley*(θ, N, P) distribution

The case when the lifetime variables are distributed *Lindley*(θ), ie they have the probability density function (PDF)

$$\varphi(x;\theta)=\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x} \quad , \ x>0,\ \theta>0 \tag{1}$$

and cumulative distribution function (CDF)

$$\Phi(x:\theta)=1-\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x} \quad , \ x>0,\ \theta>0 \tag{2}$$

Consider n random variables $L_1,\ldots,L_n \sim> Lindley(\theta)$ iid and note

$$V=\min_{1\le i\pi n} L_i \ \text{si}\ W=\max_{1\le i\pi n} L_i \tag{3}$$

Then

a) The cumulative distribution function of the minimum random variable V is

$$\Phi_V(x;\theta,n)=P(V<x)=1-P(W\ge x)=1-(1-\Phi(x;\theta))^n \ , x>0\ , \ \theta>0 \tag{4}$$

and the density probability function of random variable V is,

$$\varphi_V(x;\theta,n)=n\varphi(x;\theta)(1-\Phi(x;\theta))^{n-1} \ , x>0\ , \ \theta>0 \tag{5}$$

b) The cumulative distribution function of the maximum random variable W is

$$\Phi_W(x;\theta,n)=P(W<x)=(\Phi(x;\theta))^n \ , x>0\ , \ \theta>0 \tag{6}$$

and the density probability function of random variable W is,

$$\varphi_W(x;\theta,n)=n\varphi(x;\theta)(\Phi(x;\theta))^{n-1} \ , x>0\ , \ \theta>0 \tag{7}$$

If random variable L represents the life of a component of a parallel-connected system with the same operating characteristics, variables V and W are used to determine the reliability of the multi-component system.

Consider now a random variable N whose distribution is truncated binomial [4], denoted $Bin(n, p)$, ie probability function

$$P(N=k)=\frac{C_n^k p^k q^{n-k}}{1-q^n} \ , \ k=1,2,\ldots,n \ , \ 0<p<1 \ , \ q=1-p \tag{8}$$

Suppose that n is a sample of the random variable $N \sim> Bin(n, p)$. Then we will have:

i) the probability density function of random variable V

$$f_{V\_B\_L}(x;\theta,n,p)=\sum_{k=1}^{n}P(N=k)\varphi_v(x;\theta,n)=\sum_{k=1}^{n}\frac{C_n^k p^k q^{n-k}}{1-q^n}k\,\varphi(x;\theta)(1-\Phi(x;\theta))^{k-1} \tag{9}$$

After evaluation we obtain the expression

$$f_{V\_B\_L}(x;\theta,n,p)=\frac{n\,p}{1-q^n}\varphi(x;\theta)(1-p\,\Phi(x;\theta))^{n-1}$$

$$=\frac{n\,p}{1-q^n}\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}\left(q\,p\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)^{n-1} \ , \ q=1-p \tag{10}$$

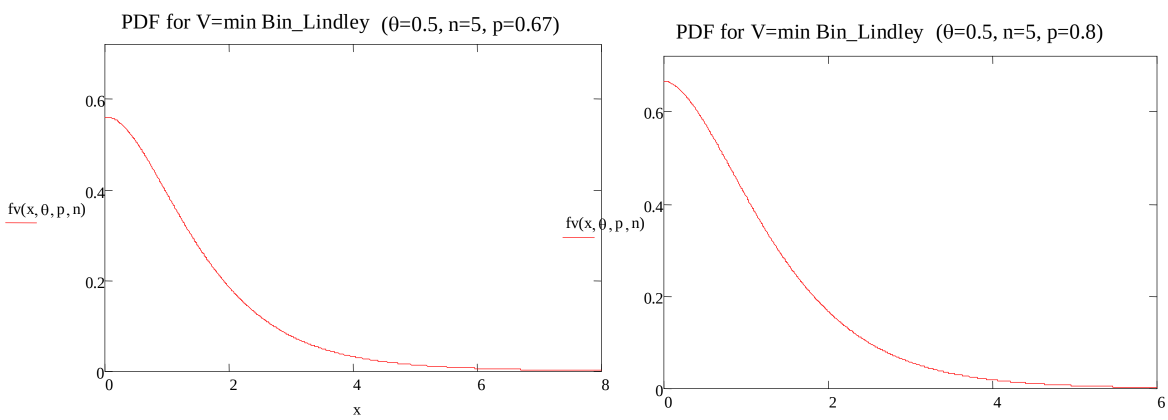

FIG. 1. PDF for random variable V with different parameters

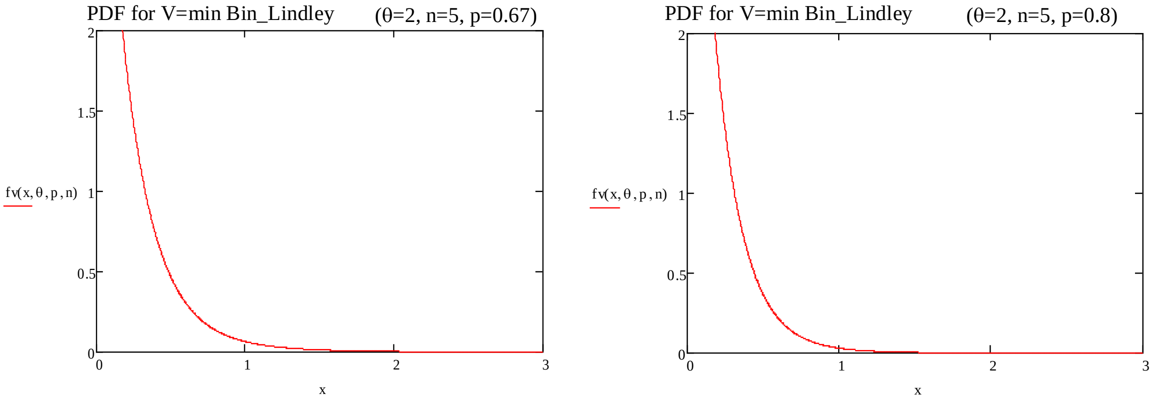

FIG. 2. PDF for random variable V with different parameters

and for the cumulative distribution function

$$F_{V\_B\_L}(x;\theta,n,p)=\int_0^x f_{V\_B\_L}(t;\theta,n,p)dt=\frac{1}{1-q^n}\left\{1-\left(1-p\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)^n\right\}=$$
$$=\frac{1}{1-q^n}\left\{1-\left(q+p\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)^n\right\},\ q=1-p \tag{11}$$

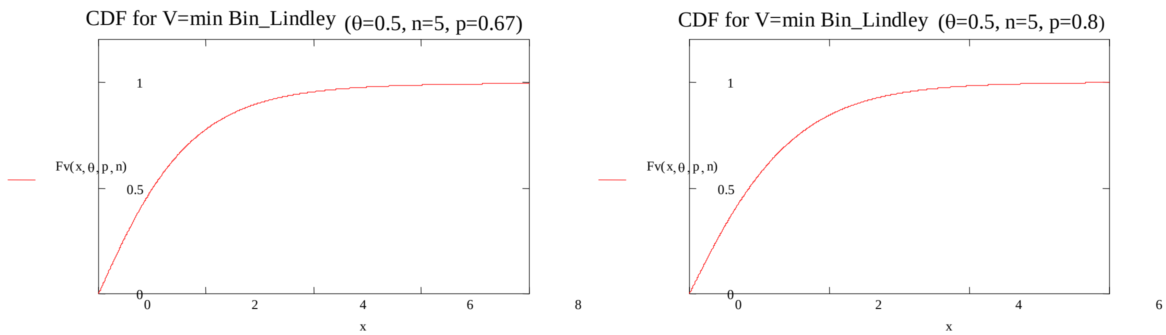

FIG. 3. CDF for random variable V with different parameters

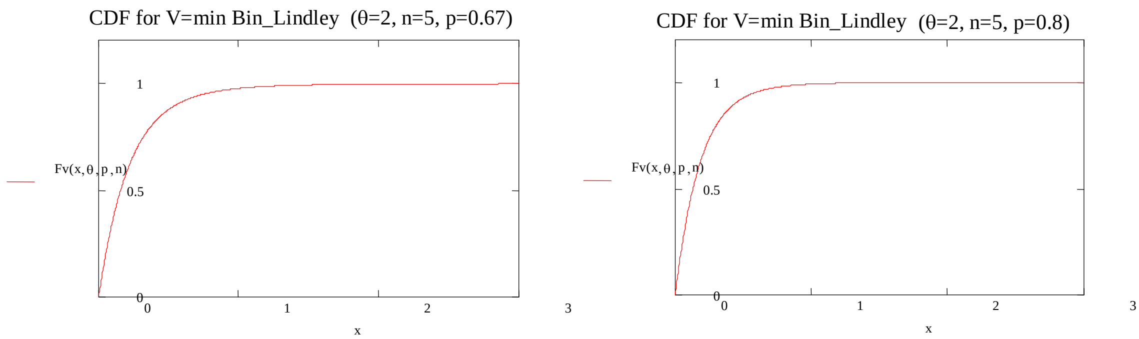

FIG. 4. CDF for random variable V with different parameters

The analogue calculations lead to the following expressions for the probability density function and the distribution function of the random variable *W*

$$f_{W\_B\_L}(x;\theta,n,p)=\frac{np}{1-q^n}\varphi(x;\theta)\left(q+p\Phi(x;\theta)\right)^{k-1}=$$
$$=\frac{np}{1-q^n}\frac{\theta^2}{1+\theta}(1+x)e^{-\theta x}\left(1-p\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)^{n-1} \tag{12}$$

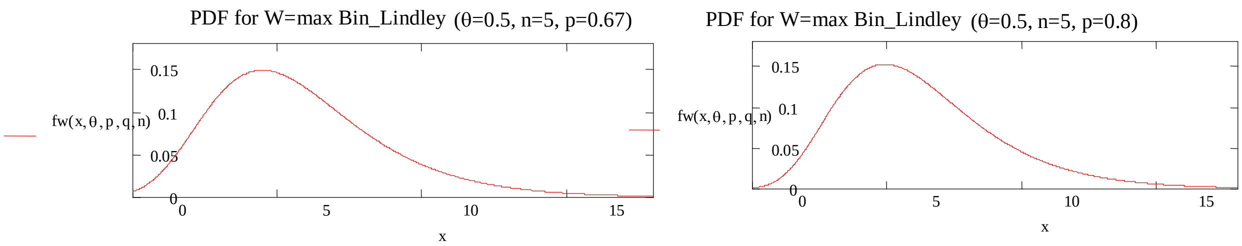

FIG. 5. PDF for random variable W with different parameters

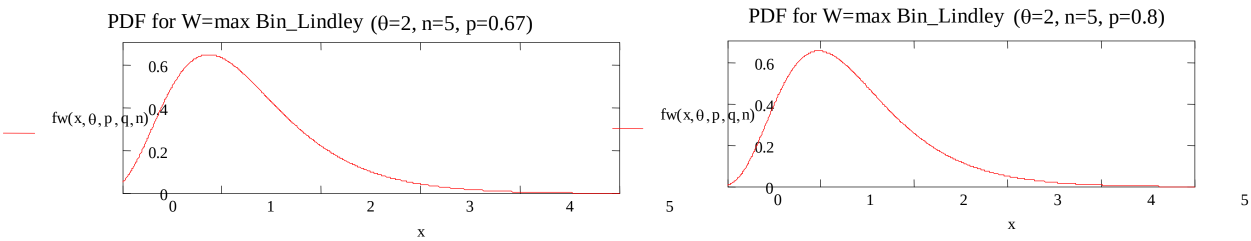

FIG. 6. PDF for random variable W with different parameters

Respectively

$$F_{W\_B\_L}(x;\theta,n,p)=\int_0^x f_{W\_B\_L}(t;\theta,n,p)dt=\frac{1}{1-q^n}\left\{\left(p\Phi(x;\theta)+q\right)^n-q^n\right\}=$$
$$=\frac{1}{1-q^n}\left\{\left(1-p\frac{1+\theta+\theta x}{1+\theta}e^{-\theta x}\right)^n-q^n\right\} \qquad (13)$$

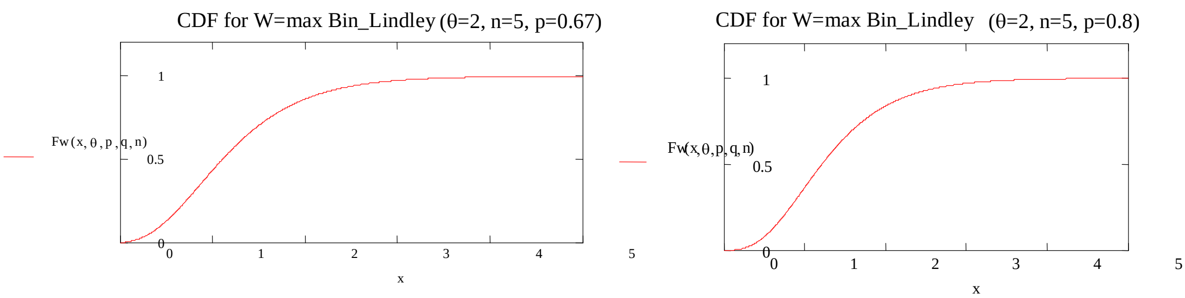

FIG. 7. CDF for random variable W with different parameters

2.2. Kemp- exponential distribution *Kemp_Exp*(*θ*, *λ*)

Consider the n variables $L_1,\ldots,L_n \sim> Exp(\lambda)$ iid having the exponential probability density function

$$\varphi(x;\lambda)=\lambda e^{-\lambda x}\ ,\ \ x>0\ ,\ \ \lambda>0 \qquad (14)$$

and the cumulative distribution function

$$\Phi(x;\lambda)=1-e^{-\lambda x}\ ,\ \ x>0\ ,\ \ \lambda>0 \qquad (15)$$

and N is a random variable having the *Kemp*(θ) distribution, i.e.

$$P(N=k)=-\frac{\theta^k}{k\ln(1-\theta)},\ \ k=1,2,\ldots,\ \ 0<\theta<1 \qquad (16)$$

With the notations above, relations (3), (4) - (7) become

a) The cumulative distribution function of the random variable V is

$$\Phi_V(x;\lambda)=P(V<x)=1-P(W\geq x)=1-(1-\Phi(x;\lambda))^n = \\ =1-e^{-\lambda n x},\ x>0,\ \theta>0 \tag{17}$$

and the probability density function of the random variable V,

$$\varphi_V(x;\lambda)=n\lambda e^{-\lambda x}\left(\lambda e^{-\lambda x}\right)^{n-1}=n\lambda e^{-\lambda n x},\ x>0,\ \theta>0 \tag{18}$$

b) The cumulative distribution function of the maximum random variable W is

$$\Phi_W(x;\lambda)=P(W<x)=(\Phi(x;\lambda))^n=\left(1-e^{-\lambda x}\right)^n,\ x>0,\ \lambda>0 \tag{19}$$

and the probability density function of the random variable W,

$$\varphi_W(x;\lambda)=n\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{n-1},\ x>0,\ \lambda>0 \tag{20}$$

Suppose now that n is a sample of the random variable $N\sim>Kemp(\theta)$. Let us consider the composition of the Kemp distribution with the distributions of the variables V and W. Then we will have:

i) The probability density function of random variable V

$$f_{V\_K\_E}(x;\lambda,\theta)=\sum_{k=1}^{\infty}\left(-\frac{\theta^k}{k\log(1-\theta)}\right)k\,\lambda\, e^{-\lambda a}(\lambda\, e^{-\lambda x})^{k-1}= \\ =-\frac{1}{\ln(1-\theta)}\sum_{k=1}^{\infty}\lambda\, e^{-\lambda x}\left(\lambda\, e^{-\lambda x}\right)^{k-1}\theta^k= \\ =-\frac{1}{\ln(1-\theta)}\sum_{k=1}^{\infty}\left(\theta\lambda\, e^{-\lambda x}\right)^k=-\frac{\theta\lambda}{\ln(1-\theta)}\left(\frac{e^{-\lambda x}}{1-\theta\, e^{-\lambda x}}\right),\ \ for\ \left(\theta\, e^{-\lambda x}\right)<1 \tag{21}$$

condition fulfilled because $0<\theta<1$, $\lambda>0$, $x>0$, $\theta<1<e^{\lambda x}$ and for the cumulative distribution function

$$F_{V\_K\_E}(x;\lambda,\theta)=\int_0^x f_{V\_K\_E}(t;\lambda,\theta)dt=\int_0^x\left(-\frac{\lambda\,\theta\, e^{-\lambda t}}{\ln(1-\theta)}\frac{1}{1-\theta\, e^{-\lambda t}}\right)dt= \\ =\left[-\frac{1}{\ln(1-\theta)}\ln\left(1-\theta\, e^{-\lambda t}\right)\right]\Big|_0^x=-\frac{1}{\ln(1-\theta)}\left[\ln\left(1-\theta\, e^{-\lambda x}\right)-\ln\left(1-\theta\, e^{-\lambda 0}\right)\right]= \\ =1-\frac{\ln\left(1-\theta\, e^{-\lambda x}\right)}{\ln(1-\theta)} \tag{22}$$

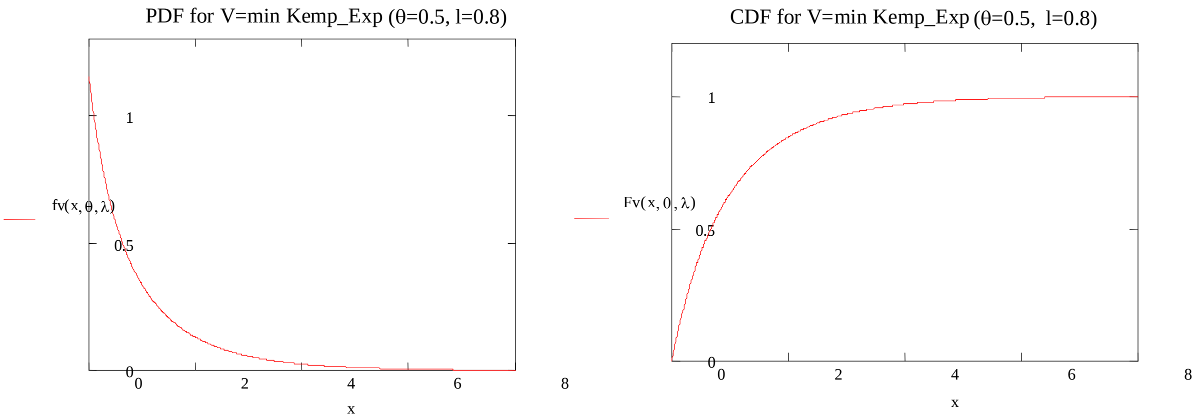

FIG. 8. PDF and CDF for random variable V with different parameters

ii) Similar calculations lead to the following expressions for the of the probability density function and the cumulative distribution function of the random variable W

$$f_{W\_K\_E}(x;\lambda,\theta)=\sum_{k=1}^{\infty}k\,\lambda\,e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{k-1}\left(-\frac{\theta^k}{k\ln(1-\theta)}\right)=$$
$$=-\frac{\theta\,\lambda\,e^{-\lambda x}}{\ln(1-\theta)}\sum_{k=1}^{\infty}\left(\theta\left(1-e^{-\lambda x}\right)\right)^{k-1}=-\frac{\theta\,\lambda}{\ln(1-\theta)}\left(\frac{e^{-\lambda x}}{1-\theta\left(1-e^{-\lambda x}\right)}\right) \tag{23}$$

respectively

$$F_{W\_B\_L}(x;\lambda,\theta)=\int_0^x\frac{\theta\,\lambda\,e^{-\lambda t}}{\ln(1-\theta)}\frac{1}{1-\theta\left(1-e^{-\lambda x}\right)}dt=\frac{1}{\ln(1-\theta)}\ln\left(1-\theta\left(1-e^{-\lambda t}\right)\right)\Big|_0^x=$$
$$=\frac{1}{\ln(1-\theta)}\left[\ln\left(1-\theta\left(1-e^{-\lambda x}\right)\right)-\ln\left(1-\theta\left(1-e^{-\lambda 0}\right)\right)\right]=\frac{\ln\left(1-\theta\left(1-e^{-\lambda x}\right)\right)}{\ln(1-\theta)} \tag{24}$$

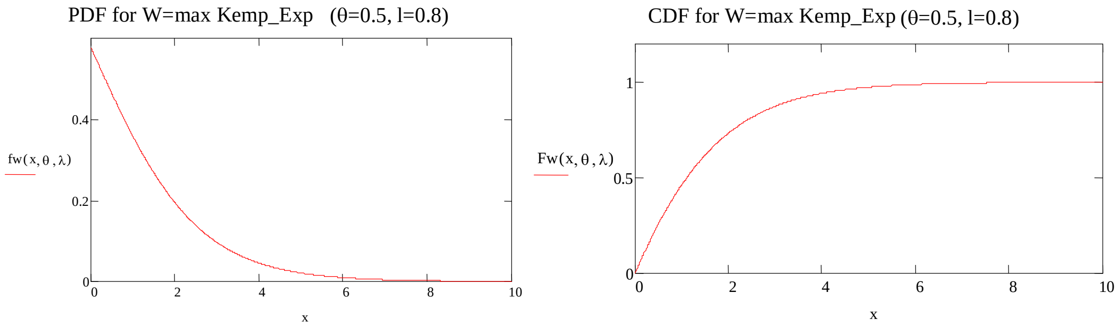

FIG. 9. PDF and CDF for random variable V with different parameters

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2.3. Zipf-exponential distribution *Zipf_Exp*(α, *N*, λ)

Consider, like in the previous case, *n* random variables $L_1,\dots,L_n \sim> Exp(\lambda)$ iid and *N* a random variable having the *Zipf*(α, *n*) distribution, ie its probability function is given by

$$P(N=k)=\frac{1}{k^{\alpha}\sum_{i=1}^{n}\frac{1}{i^{\alpha}}},\quad k=1,2,\dots n,\quad \alpha\geq 0 \tag{25}$$

If we note $H(\alpha;n)=\sum_{i=1}^{n}\frac{1}{i^{\alpha}}$ then the cumulative distribution function becomes

$$F(x)=P(N\leq x)=\frac{H_{x,\alpha}}{H_{n,\alpha}},\quad x=1,2,\dots,n\,;\,\alpha\geq 0\,, \tag{26}$$

Suppose *N* is a sample of the random variable distributed *Zipf*(α, *n*). Then, similar to relations (3), (4) - (7) we will have:

i) probability density function of random variable *V*

$$f_{V\_Z\_E}(x;\lambda,\alpha,n)=\sum_{k=1}^{n}\frac{1}{H_{\alpha,n}\,k^{\alpha}}k\lambda e^{-\lambda k x}=\frac{\lambda}{H_{\alpha,n}}\sum_{k=1}^{n}\frac{e^{-\lambda k x}}{k^{\alpha-1}} \tag{27}$$

and for the cumulative distribution function

$$F_{V\_Z\_E}(x;\lambda,\alpha,n)=\frac{1}{H_{\alpha,n}}\left(\sum_{k=1}^{n}\frac{1}{k^{\alpha}}\left(1-e^{-\lambda k x}\right)\right) \tag{28}$$

In the following figure are represented the graphs of the probability density function and the cumulative distribution function of the random variable $V \sim> Zipf\_Exp(\alpha,n,\lambda)$ for the parameter values $\alpha = 3,\ n = 5,\ \lambda = 0.5$

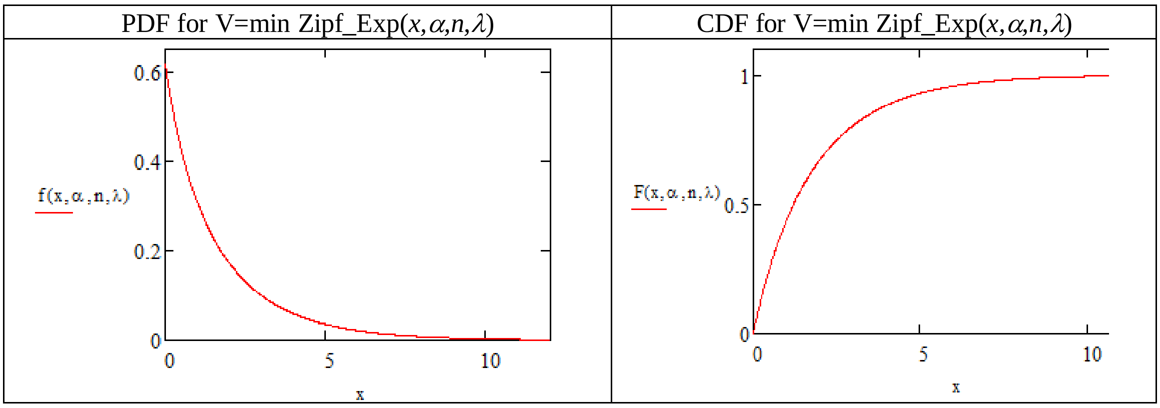

FIG. 10. PDF and CDF for random variable V with above parameters

Analogue calculations lead to the following expressions for the probability density function and the cumulative distribution function of the random variable *W*

$$f_{W\_Z\_E}(x;\lambda,\alpha,n) = \frac{1}{H_{n,\alpha}} \sum_{k=1}^{\infty} \frac{1}{k^{\alpha}} k\,\lambda \left(1 - e^{-\lambda x}\right)^{k-1} \tag{29}$$

respectively

$$F_{W\_Z\_E}(x;\lambda,\alpha,n) = \frac{1}{H_{\alpha,n}} \left(\sum_{k=1}^{n} \frac{1}{k^{\alpha}} \left(1 - e^{-\lambda x}\right)^{k} \right) \tag{30}$$

The following figures show the graphs of the probability density function and the cumulative distribution function of the random variable $W \sim> Zipf\_Exp(\alpha,n,\lambda)$ for values of parameters $\alpha = 3,\ n = 5,\ \lambda = 0.5$

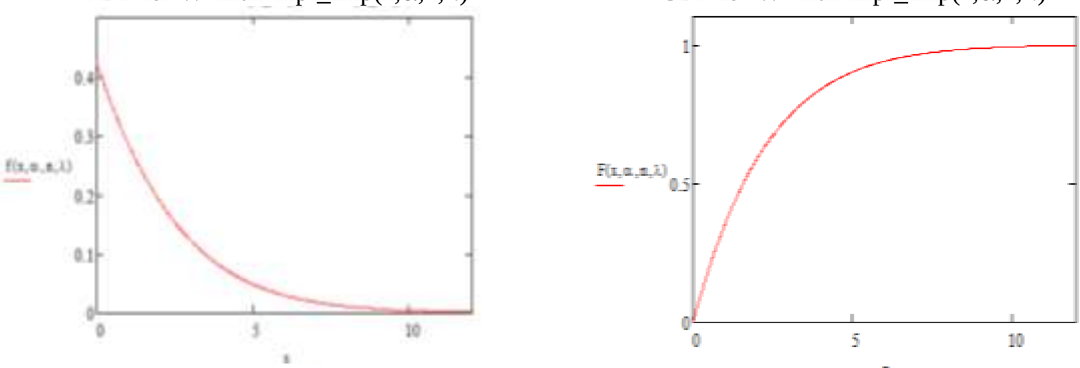

FIG. 11. PDF and CDF for random variable W with above parameters

3. NUMERICAL SIMULATION OF RANDOM VARIABLES *V / W* BY MEANS OF THE INVERSE METHOD

The following theorem is the basis of the inverse method.

Theorem. [9]

If $U \sim> \mathcal{U}(0,1)$ *then the random variable* $\mathcal{F}^{-1}(U)$ *has the cumulative distribution function* $\mathcal{F}$.

Prove. $(\forall)\, u \in (0, 1)$ and $(\forall)\, x \in \mathcal{F}^{-1}([0, 1])$, the generalized inverse function checks $\mathcal{F}^{-1}(\mathcal{F}(x)) \le x$ and $\mathcal{F}(\mathcal{F}^{-1}(u)) \ge u$. Then $\{(u,x) | \mathcal{F}^{-1}(u) \le x\} = \{(u,x) | \mathcal{F}(x) \ge u\}$ and $P(\mathcal{F}^{-1}(U) \le x) = P(U \le \mathcal{F}(x)) = \mathcal{F}(x)$.□

3.1. The case of the Bin_Lindley*(θ,N,P)* distribution

i) Numerical simulation of the random variable V

Let $V \sim> F_{V\_B\_L}(x;\theta,n,p)$ and $U \sim> \mathcal{U}(0,1)$ be two independent random variable, then the above theorem is obtained $F_{V\_B\_L}(x;\theta,n,p) = U$, ie $V = \frac{X}{\theta}$, where X is the solution of the following equation

$$\frac{1+\theta+X}{e^X} - \frac{\left(\left(1-U(1-q^n)\right)^{\frac{1}{n}} - q\right)(1+\theta)}{p} = 0 \tag{31}$$

ii) Numerical simulation of the random variable W

Let $W \sim> F_{W\_B\_L}(x;\theta,n,p)$ and $U \sim> \mathcal{U}(0,1)$, then, from the above theorem is obtained $F_{W\_B\_L}(x;\theta,n,p) = U$, that is $W = \frac{X}{\theta}$, where X is the solution of the equation

$$\frac{1+\theta+X}{e^X} - \frac{1-\left(U(1-q^n)+q^n\right)^{\frac{1}{n}}}{p}(1+\theta) = 0 \tag{32}$$

Table 1 summarizes the theoretical and empirical mean and variance obtained for 10000 simulated variables for parameter values.

Table 1. The theoretical and empirical mean and variance

Variable	Mean		Variance	
	Theoretical	Empirical	Theoretical	Empirical
V	0.2397	0.238	0.07411	0.07059
W	1.22824	1.22641	0.52662	0.5219

3.2. The case of the *Kemp_Exp*(*θ, λ*) distribution

i) Numerical simulation of the random variable V

Let $V \sim> F_{V\_K\_E}(x;\theta,\lambda)$ and $U \sim> \mathcal{U}(0,1)$, then, from the above theorem is obtained

$$F_{V\_K\_E}(x;\theta,\lambda) = U \text{ ie } V = \ln\left(\frac{\theta}{1-(1-\theta)^U}\right)^{\frac{1}{\lambda}}$$

ii) Numerical simulation of the random variable W

Let $W \sim> F_{W\_K\_E}(x;\theta,\lambda)$ and $U \sim> \mathcal{U}(0,1)$, then, from the above theorem is obtained

$$F_{W\_K\_E}(x;\theta,\lambda) = U \text{ ie } W = \ln\left(\frac{\theta}{\theta-1-(1-\theta)^U}\right)^{\frac{1}{\lambda}}$$

Table 2 shows the theoretical mean and variance and the empirical mean and variance obtained for 10000 simulated variables for parameter values.

Table 2. The theoretical and empirical mean and variance

Variable	Mean		Variance	
	Theoretical	Empirical	Theoretical	Empirical
V	1.04999	1.04248	1.3195	1.33747
W	1.48321	1.48257	1.86462	1.86015

3.3. The case of the *Zipf_Exp(α, N, λ)* distribution

The inverse simulation of the random variable V ~> Zipf_Exp (α, n, λ) returns to solving by an approximate method of the equation

$$\frac{1}{H_{\alpha,n}}\left(\sum_{k=1}^{n}\frac{1}{k^{\alpha}}\left(1-e^{-\lambda k x}\right)\right)=U \tag{33}$$

where $U \sim> \mathcal{U}(0,1)$.

For simulation by the inverse method of the random variable W ~> Zipf_Exp (α, n, λ), it is necessary to solve by an approximate method the equation

$$\frac{1}{H_{\alpha,n}}\left(\sum_{k=1}^{n}\frac{1}{k^{\alpha}}\left(1-e^{-\lambda x}\right)^{k}\right)=U \tag{34}$$

where $U \sim> \mathcal{U}(0,1)$.

For the 10000 simulated values and for the parameters values $\alpha=3$, $n=5$, $\lambda=0.5$ the theoretical and empirical mean and variance are presented in Table 3.

Table 3. The the theoretical and empirical mean and variance for α=3, n=5 , λ=0.5

Variable	Mean		Variance	
	Theoretical	Empirical	Theoretical	Empirical
V	1.82236	1.80352	3.67365	3.60838
W	2.20336	2.28037	4.44255	4.59143

4. NUMERICAL SIMULATION OF RANDOM VARIABLES V / W STARTING FROM THE DEFINITION OF THESE VARIABLES

4.1. The case of the *Bin_Lindley(θ, N, P)* distribution

We consider $U \sim> \mathcal{U}(0,1)$ and $H(X,\theta,U)=e^{X}-\frac{1+\theta+X}{(1+\theta)U}$.

If X is the solution of the equation $H(X,\theta,U)=0$, taking $x=\frac{X}{\theta}$, $\theta\neq 0$, we have a sample of the random variable *Lindley*(θ).

We generate the random variable *Lindley*(θ) by inverse method with the following algorithm.

The Lindley(θ, N) *algorithm*

P0. Input θ - distribution parameter, N - sample volume;

P1. For k: = 1; N

Generate $U \sim> \mathcal{U}(0,1)$

T:=(U>0)$\wedge$ $\left(H(0,\theta,U)\cdot H(30,\theta,U)<0\right)$);

If T = *true* then

If $x\neq 0$ then $L_k := \frac{X}{\theta}$;

P2. Returns L; Stop!

Algorithm for simulation of random variables V / W distributed Bin_Lindley(θ, n, p)

P0. Input: (*θ*, *n*, *p*), *N*-volume of the sample;

P1. For *i*: = 1; *N*

Generate $m \sim> Bin(n, p)$

while m <1 Generate $U \sim> \mathcal{U}(0,1)$, Generate $m \sim> Bin(n, p)$;

L: = Lindley (*θ*, *m*)

$V_i := \min(L_1,...,L_m)$, $W_i := \max(L_1,...,L_m)$;

P3. Returns *V*, *W*; Stop!

Applying the algorithm for *n* = 5, *p* = 0.67, *θ* = 2, for a sample of 10000 simulated values, the results from Table 4 are obtained.

Table 4. The theoretical and empirical mean and variance for *n* = 5, *p* = 0.67, *θ* = 2

Variable	Mean		Variance	
	Theoretical	Empirical	Theoretical	Empirical
V	0.2397	0.24447	0.07411	0.07848
W	1.22824	1.2385	0.52662	0.52374

4.2. The case of the *Kemp_Exp*(*θ*, *λ*) distribution

To simulate the random variable *Kemp*(*θ*) we can use the composition method [1].

Let $Y \sim> F_Y(y) = \dfrac{\ln(1-y)}{\ln(1-\theta)}$, $0<\theta<1$, $0<y<\theta$ and $N \sim> Kemp(\theta)$ then, $P(N=k|Y=y) = (1-y)y^{k-1}$, $k = 1,2,...$that is, the distribution of the random variable *N* conditioned by *Y* = *y* is a sample of the truncated geometric random variable *Geom*(*y*).

Prove.The distribution density of the variable *Y* is

$$f_Y(y) = \frac{-1}{1-y}\frac{1}{\ln(1-\theta)} , \ 0<\theta<1 , \ 0<y<\theta$$

Then

$$P(N=k) = \int_0^\theta P(N=k|Y=y)\, f_Y(y)dy = -\frac{1}{\ln(1-\theta)}\int_0^\theta y^{k-1}dy = -\frac{1}{\ln(1-\theta)}\frac{1}{k}y^k\Big|_0^\theta = -\frac{\theta^k}{k\ln(1-\theta)}$$

which means that *N* is a sample of the random variable *Kemp*(*θ*).

To simulate the Geom (*y*) geometric random variable we use the inverse method as follows:

$$P(N=k) = p\,q^{k-1} \ , \ k = 1,2,...$$

and $F(n) = P(N<n) = \sum_{k=1}^{n-1} p\,q^{k-1} = 1-q^n$, $F(n) = U$, $U \sim> \mathcal{U}(0,1)$, ie $n = \left[\dfrac{\ln U}{\ln q}\right]$.

In this case $q = y$.

Algorithm for simulating the Kemp(θ) variable by the composition method

P0. Input: *θ*, *N*-volume of the sample

P1. For *i*: = 1; *N*

Generate $U \sim> \mathcal{U}(0,1)$

$y := 1-(1-\theta)^U$

Generate $U \sim> \mathcal{U}(0,1)$

If $(y \cdot U \neq 0) \wedge (\theta \geq y)$ then $K_i := \left[\dfrac{\ln U}{\ln y} + 0.5\right]$

P2. Returns *K*; Stop!

Algorithm for simulating V / W variables distributed Kemp_Exp(θ, λ)

P0. Input: (θ, n, p), N-volume of the sample

P1. For i: = 1; N

Generate $m \sim> Kemp(\theta)$

For j: = 1; m

Generate $U \sim> \mathcal{U}(0,1)$

If $U \neq 1$ calculate $L_j := -\frac{1}{\lambda}\ln(U)$

$V_i := \min(L_1,...,L_m)$, $W_i := \max(L_1,...,L_m)$;

P3. Returns V, W; Stop!

Applying the algorithm for θ = 0.8 and λ = 0.5, for a sample of 10000 simulated values, the results in Table 5.

Table 5. The theoretical and empirical mean and variance for θ = 0.8 and λ = 0.5

Variable	Mean		Variance	
	Theoretical	Empirical	Theoretical	Empirical
V	1.33561	0.08242	2.74246	0.00668
W	2.94505	7.44391	6.07521	6.20381

4.3. The case of the *Zipf_Exp*(α, *N*, λ) distribution

To simulate the *Zipf*(α, n) distribution we use a variant of the algorithm [10]

The Zipf_invers(α, n) algorithm

P0. Input: α, n-model parameters

j: = 0;

Generate $U \sim> \mathcal{U}(0,1)$;

P1. While $F(x) = \frac{H_{x,\alpha}}{H_{n,\alpha}} > U$ calculate j: = j + 1;

P2. Returns N: = j;Stop!

For the numerical simulation of the random variable *Zipf_Exp*(α, n, λ), starting from the definition of variables V and W, we use the following algorithm.

The Zipf_Exp_Direct algorithm

P0. Input: (α, n, λ) - model parameters, N - volume of the sample;

P1. For i: = 1; N execute

m: = *Zipf_invers*(α, n);

For j: = 1; m

Generate $U \sim> \mathcal{U}(0,1)$;

$L_j := -\frac{1}{\lambda}\ln(U)$

$V_i := \min_{1\le j\le m} L_j$, $W_i := \max_{1\le j\le m} L_j$

P2. Calculate: the sample mean and variance of V and W: $M[V]$, Var[V], M [W], Var[W];

Returns $M[V]$, Var[V], $M[W]$, Var[W];

Stop!

Table 6 shows the average and the theoretical dispersion for parameter values $\alpha = 3$, $n = 5$, $\lambda = 0.5$ for a 10,000 volume sample.

Table 6 The theoretical and empirical mean and variance for $\alpha=3$, $n=5$, $\lambda=0.5$

Variable	Mean		Variance	
	Theoretical	Empirical	Theoretical	Empirical
V	1.82236	1.80352	3.67365	3.60838
W	2.20336	2.28037	4.44255	4.59143

The numerical results of applying the two above algorithms are listed in Table 6.

5. CONCLUSIONS

In this paper we obtained three probability distributions with possible applications in the reliability of multi-component systems using the computation method (consisting of discrete distributions with continuous distributions). For these distributions we simulated 10,000 variables by the inverse method and using their definition for different parameter values and we compared the methods by considering the theoretical mean and the variance with the sampling mean and variance respectively. It can be concluded that the methods lead to good results as can be seen from the Tables 1-6.

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