DECOMPOSITION OF THE TIME SERIES AND OF SHOCKS USING THE SIMPLE FRACTIONS DECOMPOSITION AND APPLICATIONS

Daniel CIUIU

Technical University of Civil Engineering Bucharest (dciuiu@yahoo.com)

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Abstract: *In this paper we will use the decomposition of rational functions in simple fractions. The rational functions are build using the delay polynomials $\varphi(L)$ and $\theta(L)$ of an ARIMA time series.*

For decomposition of the time series X_t we use the rational fraction $\frac{\theta(L)}{\varphi(L)}$, and for the decomposition of the white noise a_t we use the rational fraction $\frac{\varphi(L)}{\theta(L)}$.

Finally, because for the decomposition of X_t we do not take into account that the roots of $\varphi(L)$ are greater than one in absolute value, we eventually multiply in the first above case $\varphi(L)$ by $(1-L)^d$ for taking into account the possible trend and by $(1-L^s)^{d_s}$ for taking into account the possible seasonal components.

Keywords: *ARMA and ARIMA time series, delay operator, delay polynomials.*

1. INTRODUCTION

The classical decomposition of time series is [3,4] in seasonal components, trend and a stationary component. The remaining stationary component, even we obtain it by removing seasonal components and trend, even we obtain it by seasonal and non-seasonal differentiation, is modeled as $AR(p)$, $MA(q)$ or $ARMA(p,q)$ time series.

For an $ARMA(p,q)$ we can write

$$\varphi(L)X_t = \theta(L)a_t, \text{ where} \tag{1}$$

$$\begin{cases} \varphi(L) = 1 - \sum_{i=1}^{p} \varphi_i L^i \\ \theta(L) = 1 - \sum_{i=1}^{q} \theta_i L^i \end{cases}. \tag{1'}$$

Using the above formula, we obtain [2,3,4]

$$\begin{cases} X_t = \frac{\theta(L)}{\varphi(L)} a_t \\ a_t = \frac{\varphi(L)}{\theta(L)} X_t \end{cases}. \tag{1''}$$

2. THE DECOMPOSITION OF TIME SERIES

For decomposition of X_t we use the decomposition in simple fractions of $\frac{\theta(L)}{\varphi(L)}$. Denote now the roots of $\varphi(L)$ $x_1, ..., x_l$ with the multiplicities $m_1, ..., m_l$.

If X_t is $ARMA(p,q)$ with $p>q$, there exists the white noise a_t such that

$$X_t = \sum_{i=1}^{l}\sum_{j=1}^{m_i}\frac{A_{ij}}{\left(1-x_i\cdot L\right)^j}a_t. \tag{2}$$

If we have $p\leq q$, the above formula becomes

$$X_t = \tilde{\theta}(L)a_t + \sum_{i=1}^{l}\sum_{j=1}^{m_i}\frac{A_{ij}}{\left(1-x_i\cdot L\right)^j}a_t, \text{ where} \tag{2'}$$

$\tilde{\theta}(L)$ is the quote of $\frac{\theta(L)}{\varphi(L)}$.

In formulae (2) and (2') the roots of $\varphi(L)$ can be complex. In this case we can group the conjugate complex roots. We obtain at denominator $\left(1-2\operatorname{Re}(z_i)+|z|^2\right)^{m_i}$, and the numerator becomes a real polynomial of degree m_i. In the case $m_i=1$ (simple complex roots), we obtain a linear numerator, and a second degree function at denominator. It results that if $p>q$, the $ARMA(p,q)$ is a sum of $AR(j)$ with $1\leq j\leq m_i$ for real x_i, and a time series similar to $ARMA(2\cdot m_i, m_i)$ for complex conjugate roots with the multiplicity m_i.All the above parts of X_t have the same white noise a_t, except multiplying by a constant. If $p=q$ we add to the above decomposition the term $\frac{\theta_p}{\varphi_p}a_t$, and if $p<q$ we add the term $\tilde{\theta}(L)a_t$, i.e. a polynomial of degree $q-p$ in lag L applied to the same white noise a_t.

If we consider the reverse in (1"), we decompose analogously a_t in terms of X_t. For forecasting we can forecast each term in the decomposition of X_t.In the above decomposition of X_t the fact that the roots of $\varphi(L)$ are in absolute value grater than one is used only for stationarity, not for decomposition. For instance, if the time series is $ARIMA(p,d,q)$ we use instead of $\varphi(L)$ $(1-L)^d\varphi(L)$. If we group the unit root and the roots of $\varphi(L)$, we obtain a decomposition in $ARIMA(0,j,0)$ with $j=\overline{1,d}$ and an $ARMA(p,q)$ time series. If we perform also the seasonal differentiation $\left(1-L^s\right)^{d_s}$, we group also the complex roots of equation $L^s=1$.

In the case of stationarizing using the removing trend by moving average method, we have to find the roots of $\sum_{j=0}^{2\cdot q}L^j-(2\cdot q+1)L^q=0$, corresponding to the differences $\hat{m}_{t-q}-X_{t-q}$, i.e the reminding stationary time series after removing the moving average of order $2\cdot q+1$, with opposite sign. Using two times the scheme of Horner, we obtain $L=1$ of multiplicity 2. The other roots are the roots of polynomial

$$\sum_{j=0}^{q-1}\frac{(j+1)(j+2)}{2}\left(L^{j}+L^{2\cdot q-j}\right)+\frac{q(q+1)}{2}L^{q}. \tag{3}$$

By multiplying $\sum_{j=0}^{2\cdot q}L^{j}-(2\cdot q+1)L^{q}=0$ by L-1, we can prove that the only multiple root is one (multiplicity is two), and we have no other root on the unit circle. Between the other $2\cdot q-2$ roots we can prove that we have at most two real roots. From the theory of symmetric polynomials, it results that mainly the other $2\cdot q-2$ roots are clustered in groups of four: L_j, $\overline{L_j}$, $\frac{1}{L_j}$ and $\frac{1}{\overline{L_j}}$. The two real roots appear if four does not divide $2\cdot q-2$, hence for even values of q. For odd values of q, these solutions are all simple and conjugated complex in the above groups of four. If we use a moving average with $q=1$, the roots are $L_1=L_2=1$. If $q=2$, the other two roots are the roots of second degree equation $L^2+3L+1=0$, having the roots $-\alpha_1^2$ and $-\alpha_2^2$, where $\alpha_j=\frac{1\pm\sqrt{5}}{2}$, from Fibonacci stream. The roots of polynomial involving moving average of order $2\cdot q+1$ with even and odd q are presented in Tables 6 and 7, Appendix A.

The following structure of solutions has not been proved, but it was checked for $q=4,6,\ldots,20$, $q=100$, $q=500$ and $q=1000$, and for $q=3,5,\ldots,19$, $q=99$, $q=499$ and $q=999$. For even values of q the real negative roots make a circular crown with the radius the absolute values (the other roots have the absolute values between the two radius). The minimum absolute value (that of the real root ≥ -1) increases from 0.38197 for q=2 to 0.806351 for q=20, 0.94207 for q=100, 0.98493 for q=500 and 0.99174 for q=1000. For the minimum argument of complex roots expressed in degrees, the value $\frac{360}{\arg\min\cdot q}$ decreases from 1.08145 for q=4 to 1.0223052 for q=20, 1.0048508 for q=100, 1.0009924 for q=500, and 1.0004979 for q=1000. For odd values of q we have not real roots (all roots are complex in above groups of four). But the minimum absolute value is also increasing on q: from 0.47568 for q=3 to 0.79966 for q=19, 0.94161 for q=99, 0.9849 for q=499, and 0.99174 for q=999. The expression $\frac{180}{\arg\min\cdot q}$ decreases from 1.102567 for q=3 to 1.023379 for q=19, 1.0048986 for q=99, 1.0009944 for q=499, and 1.0004984 for q=999.

In the case of exponential smooth we multiply $\varphi(L)$ by $\frac{1-L}{1-\alpha L}$, where α is the ratio of decreasing the weights of exponential smooth. If α is the inverse of a root of $\varphi(L)$, we divide φ by $1-\alpha L$, otherwise we multiply θ by $1-\alpha L$. Of course, in both cases we multiply φ by $1-L$.

The effective decomposition of X_t is made starting from the moment t just before the first computed a_t. For instance, in an AR(p) model first t is p. We decompose this first X_t in $X_t^{(1)},\ldots,X_t^{(p)}$, and $b_t^{(j)}$ are the drifted white noises from the initially one multiplied by the constants from fractions decomposition. It results a linear regression with the coefficients $X_t^{(1)},\ldots,X_t^{(p)}$. The white noise starts in decomposition of X_t by multiplied by the constants, and b_t is decomposed in MA(1) like white noises ($X_t^{(j)}$ is revertible, but not necessary stationary).

3. APPLICATION

Consider the CPI (Consumer Prices Index) from Buletinul Institutului National de Statistică [6] expressed in percentage of current month related to previous, in the period January 1991 - February 2017.

We want to express the time series X_t as in ARIMA model, and next to decompose the time series X_t and the white noise a_t. First we notice that, using the Dickey - Fuller unit root test [2] that the time series is not stationary, but the difference $\Delta X_t = X_t - X_{t-1}$ is. In the case of AR (p) and MA(q)with $p,q=\overline{0,5}$, not both zero, the representations of X_t are presented in Table 1, that follows.

Table 1 – Representations of X_t for AR (p) and MA(q) time series

p|q	AR(p)	MA(q)
1	$-0.38675\,X_{t-1}+a_t$	$a_t-0.38675\,a_{t-1}$
2	$-0.48401\,X_{t-1}-0.25149\,X_{t-2}+a_t$	$a_t-0.48401\,a_{t-1}-0.0643\,a_{t-2}$
3	$-0.52059\,X_{t-1}-0.32189\,X_{t-2}-0.14546\,X_{t-3}+a_t$	$a_t-0.52059\,a_{t-1}-0.06992\,a_{t-2}+0.0125\,a_{t-3}$
4	$-0.53818\,X_{t-1}-0.36081\,X_{t-2}-0.20841\,X_{t-3}-0.12091\,X_{t-4}+a_t$	$a_t-0.53818\,a_{t-1}-0.08064\,a_{t-2}+0.00386\,a_{t-3}-0.02384\,a_{t-4}$
5	$-0.55532\,X_{t-1}-0.39035\,X_{t-2}-0.25955\,X_{t-3}-0.19719\,X_{t-4}-0.14174\,X_{t-5}+a_t$	$a_t-0.55532\,a_{t-1}-0.09149\,a_{t-2}-0.01155\,a_{t-3}-0.04642\,a_{t-4}-0.04043\,a_{t-5}$

In the AR(p) case we obtain the following results for $p=\overline{1,5}$.

Table 2 – Decomposition of X_t for ARIMA(p,1,0) time series

p	Simple fractions for AR(p)	Simple fractions for X_t
1		$\frac{0.72111}{1-L}+\frac{0.27819}{1+0.38675L}$
2	$\frac{0.5+0.27549i}{1+(0.24201-0.43923i)L}+\frac{0.5-0.27549i}{1+(0.24201+0.43923i)L}$	$\frac{0.5762}{1-L}+\frac{0.21899-0.48207i}{1+(0.24201-0.43923i)L}+\frac{0.21899+0.48207i}{1+(0.24201+0.43923i)L}$
3	$\frac{0.44899}{1+0.48059L}+\frac{0.2755+0.26718i}{1+(0.02-0.54979i)L}+\frac{0.2755-0.26718i}{1+(0.02+0.54979i)L}$	$\frac{0.5762}{1-L}+\frac{0.21899-0.48207i}{1+(0.24201-0.43923i)L}+\frac{0.21899+0.48207i}{1+(0.24201+0.43923i)L}$
4	$\frac{0.15478+0.19809i}{1-(0.18476+0.57486i)L}+\frac{0.15478-0.19809i}{1-(0.18476-0.57486i)L}+\frac{0.34525+0.07649i}{1+(0.45385-0.3544i)L}+\frac{0.34525-0.07649i}{1+(0.45385+0.3544i)L}$	$\frac{0.44877}{1-L}+\frac{0.1424-0.0536i}{1-(0.18476+0.57486i)L}+\frac{0.1424+0.0536i}{1-(0.18476-0.57486i)L}+\frac{0.13321-0.02782i}{1+(0.45385-0.3544i)L}+\frac{0.13321+0.02782i}{1+(0.45385+0.3544i)L}$
5	$\frac{0.28006}{1+0.64862L}+\frac{0.10794+0.13626i}{1-(0.37203+0.58036i)L}+\frac{0.10794-0.13626i}{1-(0.37203-0.58036i)L}+\frac{0.25203-0.11077i}{1+(0.32538+0.59495i)L}+\frac{0.25203+0.11077i}{1+(0.32538-0.59495i)L}$	$\frac{0.3943}{1-L}+\frac{0.10759}{1+0.64862L}+\frac{0.12175-0.06692i}{1-(0.37203+0.58036i)L}+\frac{0.12175+0.06692i}{1-(0.37203-0.58036i)L}+\frac{0.12731+0.02947i}{1+(0.32538+0.59495i)L}+\frac{0.12731-0.02947i}{1+(0.32538-0.59495i)L}$

In the above table, for instance in the AR(3) model X_t is decomposed in three AR(1) time series with the polynomial $\varphi_1(L)=1+0.48059L$, $\varphi_2(L)=1+(0.02-0.54979i)L$ and $\varphi_3(L)=1+(0.02+0.54979i)L$, and the white noises the white noise a_t of X_t multiplied by 0.44899, 0.2755+0.26178 i, respectively 0.44899, 0.2755-0.26178 i.

If we consider the non-zero expectation case, the above white noise a_t is substituted by the drifted noise $b_t = a_t + \varphi(1)\cdot m$, where $\varphi(L) = 1 + 0.52059L + 0.32189L^2 + 0.14546L^3$, according Table 1, hence φ(1)=1.98794. Because m=-0.04757 it results that the drift is -0.09457, hence we subtract from a_t the value 0.09457. Using this b_t we obtain the same three components for initial time series, but b_t is multiplied by other coefficients: 0.14574, 0.17561-0.0486 i, and 0.17561+0.0486 i. In addition, corresponding to the root L=1 in the ARIMA case, we have an ARIMA(0,1,0) component Y_t such that the difference is b_t multiplied by 0.50303. The decompositions of initial time series X_t and of the drifted noise b_t for ARIMA(0,1,q) are presented in the following table.

Table 3 – Decomposition of X_t a_t for ARIMA(0,1,q) time series

q	Simple fractions for X_t	Simple fractions for a_t
1	$0.38675 + \frac{0.61325}{1-L}$	$2.58565 - \frac{1.58565}{1-0.38675L}$
2	$0.54831 + 0.0643L + \frac{0.45169}{1-L}$	$-\frac{0.5812}{1-0.59253L} + \frac{1.5812}{1+0.10852L}$
3	$0.57801 + 0.05742L - 0.0125L^2 + \frac{0.42199}{1-L}$	$-\frac{0.6125}{1-0.60223L} + \frac{0.9556}{1+0.19056L} + \frac{0.657}{1-0.10892L}$
4	$0.6388 + 0.10062L + 0.01998L^2 + 0.02384L^3 + \frac{0.3612}{1-L}$	$-\frac{0.2872}{1-0.7105L} + \frac{0.5433}{1+0.33985L} + \frac{0.3719+0.2236i}{1-(0.08377+0.30284i)L} + \frac{0.3719-0.2236i}{1-(0.08377-0.30284i)L}$
5	$0.74521 + 0.18989L + 0.0984L^2 + 0.08685L^3 + 0.04043L^4 + \frac{0.25479}{1-L}$	$-\frac{0.1038}{1-0.84007L} + \frac{0.2381+0.196i}{1-(0.21154+0.47124i)L} + \frac{0.2381-0.196i}{1-(0.21154-0.47124i)L} + \frac{0.3138+0.1094i}{1+(0.35392-0.23478i)L} + \frac{0.3138-0.1094i}{1+(0.35392+0.23478i)L}$

For instance, the decomposition of ARIMA(0,1,3) is X_t=0.57801 b_t+0.05742 b_{t-1}+0.0125 b_{t-2}+Y_t, where Y_t is an ARIMA(0,1,0) time series with difference equal to 0.42199*b_t.

In the following we consider the model ARIMA(p,1,q), where the size of the ARMA model, the value of p+q, is constant. The values of $\varphi(L)$ for $p = 4,3,2,1$ are $1 - 0.05275L - 0.07812L^2 + 0.04559L^3 + 0.11309L^4$, $1 - 0.0486L - 0.48217L^2 - 0.16583L^3$, $1 + 0.55701L - 0.10944L^2$, respectively $1 + 0.92335L$. The corresponding values of $\theta(L)$ are $1 - 0.53855L$, $1 - 0.54042L - 0.41813L^2$, $1 + 0.07329L - 0.38463L^2 - 0.10279L^3$, and $1 + 0.61752L - 0.59303L^2 - 0.21334L^3 - 0.14483L^4$.

Consider now $p + q = 5$ with $1 \le p \le 4$. The results for decomposition of the ARMA(p,q) time series Y_t and of the initial ARIMA(p,1,q) time series X_t are presented in the following table.

Table 4 – Decomposition of X_t and Y_t for ARIMA(p,1,q) time series with p+q=5

p	Simple fractions for Y_t	Simple fractions for X_t
4	$\frac{0.38451-0.19825i}{1+\left(0.42136+0.35245i\right)L}+\frac{0.38451+0.19825i}{1+\left(0.42136-0.35245i\right)L}+$ $\frac{0.1155-0.15024i}{1-\left(0.44774-0.41748i\right)L}+\frac{0.1155+0.15024i}{1-\left(0.44774+0.41748i\right)L}$	$\frac{0.44896}{1-L}+$ $\frac{0.16224-0.00366i}{1+\left(0.42136+0.35245i\right)L}+\frac{0.16224+0.00366i}{1+\left(0.42136-0.35245i\right)L}+$ $\frac{0.11328+0.12348i}{1-\left(0.44774-0.41748i\right)L}+\frac{0.11328-0.12348i}{1-\left(0.44774+0.41748i\right)L}$
3	$\frac{0.54943+0.0823i}{1+\left(0.61666+0.29215i\right)L}+\frac{0.54943-0.0823i}{1+\left(0.61666-0.29215i\right)L}-$ $\frac{0.09886}{1-0.84791L}$	$\frac{0.13662}{1-L}+\frac{0.15613+0.0767i}{1+\left(0.61666+0.29215i\right)L}+$ $\frac{0.15613-0.0767i}{1+\left(0.61666-0.29215i\right)L}+\frac{0.55113}{1-0.84791L}$
2	$-\frac{7.64178}{1-0.15394L}+\frac{0.34688}{1+0.71095L}+8.2949+0.93924L$	$\frac{0.40473}{1-L}+\frac{1.39037}{1-0.15394L}+\frac{0.14414}{1+0.71095L}-0.93924$
1	$-\frac{0.2926}{1+0.92335L}+1.2926-0.576L-$ $0.06118L^2-0.15685L^3$	$\frac{0.34644}{1-L}-\frac{0.14047}{1+0.92335L}+0.79403+$ $0.21803L+0.15685L^2$

The corresponding decompositions of the white noise in the ARMA and ARIMA cases are presented in Table 6, that follows.

Table 5 – Decomposition of a_t for ARIMA(p,1,q) time series with p+q=5

p	Simple fractions for ARMA(p,q)	Simple fractions for X_t
4	$\frac{1.47457}{1-0.53855L}+0.42182+0.39645L+$ $0.25558L^2+0.11309L^3$	$-\frac{1.94412}{1-0.53855L}-1.58556-0.28694L-$ $0.14087L^2-0.14249L^3-0.11309L^4$
3	$\frac{0.18107}{1+0.43061L}+\frac{0.17836}{1-0.97103L}+$ $0.64057+0.3966L$	$\frac{0.60158}{1+0.43061L}-\frac{0.00532}{1-0.97103L}+0.40374-$ $0.24397L-0.3966L^2$
2	$\frac{0.117357+2.82097i}{1+\left(0.38772+0.02902i\right)L}+$ $\frac{0.117357-2.82097i}{1+\left(0.38772-0.02902i\right)L}+\frac{0.65287}{1-0.69415L}$	$-1.0647+\frac{1.17618+10.09684i}{1+\left(0.38772+0.02902i\right)L}+$ $\frac{1.17618-10.09684i}{1+\left(0.38772-0.02902i\right)L}-\frac{0.28767}{1-0.69415L}$
1	$\frac{0.20482+0.09338i}{1+\left(0.15477+0.38077i\right)L}+$ $\frac{0.20482-0.09338i}{1+\left(0.15477-0.38077i\right)L}+$ $\frac{0.6087}{1-0.78463L}-\frac{0.01835}{1+0.89667L}$	$\frac{0.60294-0.28271i}{1+\left(0.15477+0.38077i\right)L}+$ $\frac{0.60294+0.28271i}{1+\left(0.15477-0.38077i\right)L}-$ $\frac{0.16708}{1-0.78463L}-\frac{0.03881}{1+0.89667L}$

CONCLUSIONS

In [2,3,4] the decomposition of a time series in seasonal component, trend and stationary has been performed using for instance moving average. Analogously, if we use the differentiation and/ or seasonal differentiation we can group the root one and the complex unit root for seasonal differentiation. Other decompositions are performed due to economic reasons, as the decomposition of GDP in [1,5]. An open problem is if the economic decomposition can be naturally performed by grouping this paper decomposition of time series.

We have said "similar to ARMA(2*m,m)" instead of ARMA(2*m,m) in Section 2, because the roots of numerator are not necessary in absolute value greater than one. For instance, in the case of AR(5), if we add the corresponding AR(1) components $\frac{0.2755+0.26718i}{1+(0.02-0.54979i)L}$ and the conjugate, we obtain $\frac{0.21588-0.23847L}{(1+(0.02-0.54979i)L)(1+(0.02+0.54979i)L)}$, which has obviously roots for denominator greater than one in absolute value, but the numerator has the root L=0.90523! For MA(q) with $q=\overline{2,5}$ the quote of degree q-1 has in all four cases in Table 3 roots greater than one in absolute value. An open problem is if this is a rule, or it happens in our example and other ones.

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APPENDIX A. ROOTS FOR MOVING AVERAGE

Table 6 – Roots for even values of q

q	Real root ≥ -1	The other absolute values ≤ 1	The other angles in degrees
2	-0.38197		
4	-0.52031	0.55242	83.22129
6	-0.60296	0.65776; 0.61432	66.52096; 56.51432
8	-0.65882	0.72426; 0.68255; 0.66423	85.88092; 50,73123; 42.88701
10	-0.69947	0.76948; 0.73145; 0.71223; 0.70249	76.67192; 69.20316; 41.01439; 34.58262
12	-0.73058	0.8021; 0.76772; 0.74918; 0.73821; 0.73244	87.02103; 63.84469; 57.98187; 34.42691; 28.98308
14	-0.75264	0.82669; 0.79555; 0.77812; 0.76717; 0.76031; 0.75649	80.1066; 74.88039; 55.00913; 49.90537; 29.66504; 29.94881
16	-0.77539	0.84588; 0.8175; 0.80126; 0.79065; 0.78351; 0.77887; 0.77624	87.66116; 70.36931; 65.72646; 48.32439; 43.80988; 26.06156; 21.90285
18	-0.79215	0.86126; 0.83525; 0.82012; 0.81001; 0.80291; 0.79796; 0.79466; 0.79277	82.33238; 78.11198; 62.74497; 58.57341; 43.08966; 39.04454; 23.23933; 19.52106
20	-0.80635	0.87389; 0.84988; 0.83578; 0.82619; 0.81929; 0.81426; 0.81066; 0.80822; 0.80621	88.07264; 74.2845; 70.44565; 56.61257; 52.82802; 38.87899; 35.21603; 20.96897; 17.60727

Table 7 – Roots for odd values of q

q	The absolute values ≤ 1	The angles in degrees
3	0.47568	54.41846
5	0.61161; 0.57041	78.82098; 33.58896
7	0.69447; 0.65137; 0.63517	73.19168; 57.55775; 24.36856
9	0.70893; 0.68982; 0.68159; 0.74885	84.13503; 54.44701; 45.35669; 19.14011
11	0.78702; 0.75086; 0.73188; 0.72136; 0.71657	78.89304; 69.4227; 47.30958; 37.43245; 15.76617
13	0.8152; 0.7825; 0.7645; 0.7535; 0.74701; 0.74396	86.06313;67.06126; 59.0979; 40.22503; 31.86881; 13.40661
15	0.83685; 0.80714; 0.7903; 0.7795; 0.77244; 0.76812	81.68513; 74.92242; 58.32909; 51.45025; 34.99151; 27.74666; 11.66294
17	0.85397; 0.82683; 0.81115; 0.80078; 0.79364; 0.7888; 0.78577; 0.78431	87.0486; 72.27458; 66.33853; 51.6154; 45.55699; 30.96595; 24.56959; 10.32147
19	0.86786; 0.8439; 0.8283; 0.81844; 0.81442; 0.80643; 0.80294; 0.80073; 0.79966	83.3456; 78.10145; 64.81504; 59.52108; 46.29108; 40.87608; 27.7727; 22.04581; 9.25726