MODIFIED ADOMIAN DECOMPOSITION METHOD FOR SOLVING RICCATI DIFFERENTIAL EQUATIONS

Necdet BİLDİK, **Sinan DENİZ**
Celal Bayar University, Manisa/TURKEY
DOI: 10.19062/1842-9238.2015.13.3.3

Abstract: *In this study, we solve Riccati differential equations by modified Adomian decomposition method which is constructed by different orthogonal polynomials. Here, Chebyshev polynomials are used instead of Taylor polynomials to expand the source function. We see the benefits of using these expansions to get better results.*
Keywords: *Chebyshev polynomials, Adomian decomposition method, nonlinear differential equations.*

1. INTRODUCTION

Adomian decomposition method was found by George Adomian and has recently become a very well - known method in applied sciences.

The method does not need any linearization or smallness assumptions to solve the differential equations and this makes the method very effective among the other methods.

Many works have been examined in various different areas such as heat or mass transfer, nonlinear optics, incompressible fluid and gas dynamics phenomena etc [1-5].

Nonlinear differential equations arise from many important applications in physics, engineering, applied science such as damping laws, diffusion processes, transmission line phenomena, etc.

They have been solved by many different techniques [6-9].

Riccati differential equation is one of the most significant nonlinear differential equations.

They are generally used in the studies of optimal control problems.

Obtaining the exact solution of this equation is not always possible.

So, numerical methods are needed to get the approximate solution.

Traditional Adomian decomposition method is also used to solve Riccati differential equation in many different papers [10-12].

2. ADOMIAN DECOMPOSITION METHOD

In this section we give some brief and basic information about Adomian decomposition method. For much more information, we refer to [5,13,14].

Consider the differential equation

$$Ly + Ry + Ny = g(t) \tag{1}$$

where L is the highest-order derivative which is assumed to be invertible, R is a linear differential operator of less order than L, N is the nonlinear operator and g is the source term. If we apply the operator L^{-1} which is the inverse of the L to the equation (1), we get

$$L^{-1}(Ly) = y = L^{-1}(g) - L^{-1}(Ry) - L^{-1}(Ny). \tag{2}$$

Let us suppose the solution of the Eq.(1):

$$y(t) = \sum_{n=0}^{\infty} y_n(t). \tag{3}$$

Besides that the nonlinear terms is obtained by

$$Ny = \sum_{n=0}^{\infty} A_n \tag{4}$$

where A_n are Adomian polynomials which can be calculated from:

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[N\left(\sum_{i=0}^{\infty}\lambda^i t_i\right)\right]_{\lambda=0}, \quad n = 0,1,2,\ldots \tag{5}$$

Using the equations (2)-(5), we get

$$\sum_{n=0}^{\infty} y_n = f - L^{-1}\left(R\left(\sum_{n=0}^{\infty} y_n\right)\right) - L^{-1}\left(\sum_{n=0}^{\infty} A_n\right). \tag{6}$$

where f is calculated from the source term and the given condition(s) which are assumed to be prescribed. We now construct the recursive relation as :

$$\begin{aligned} y_0 &= f = \Psi_0 + L^{-1}(g(t)) \\ y_1 &= -L^{-1}R(y_0) - L^{-1}(A_0) \\ &\vdots \\ y_{k+1} &= -L^{-1}\left(R\left(y_k\right)\right) - L^{-1}\left(A_k\right), \quad k \geq 0 \end{aligned} \tag{7}$$

It can be easily said that the solution is

$$y = \lim_{m\to\infty}\left(\sum_{n=0}^{m} y_n\right) \tag{8}$$

provided that the series converges suitably.

3.MODIFIED ADOMIAN DECOMPOSITION METHOD

In this section, we give the construction of modified Adomian decomposition method by using Chebyshev polynomials. Normally, we use Taylor polynomials in calculations for Adomian decomposition method. However, as we will see that using Chebyshev polynomials yields better results than Taylor polynomials.

Generally, the source term is usually written as

$$g(t) \approx \sum_{n=0}^{m} \frac{g^n(0)}{n!} t^n . \tag{9}$$

Hosseini [15] used Chebyshev polynomials to modify the ADM by expanding:

$$g(t) \approx \sum_{n=0}^{m} a_n T_n(t) \tag{10}$$

where a_n are coefficients and T_n are Chebyshev polynomials [16,17].

In fact, there are many orthogonal polynomials as Laguerre, Legendre etc. that we can use instead of Taylor polynomials. But, Tien [18] and Mahmoudi [19] showed that these modifications are not good enough as much as Chebyshev polynomials.

4. NUMERICAL EXAMPLES

In this section, we solve two Riccati equations to illustrate the phenomena. These problems are brand new and cannot be found in the literature.

Example 1) Consider the Riccati differential equation

$$y' - \frac{1}{2}y + y^2 = e^t , y(0) = 1 \tag{11}$$

which have the exact solution $y = e^{\frac{t}{2}}$.

Solution:

We proceed according to section 2. We have

$$L = \frac{d}{dt}, R(y) = \frac{-y}{2}, Ny = F(y) = y^2$$

and $g(t) = e^t$. (12)

Constructing Adomian polynomials according to (5), we obtain:

$$\begin{aligned} A_0 &= F(y_0) = y_0^2 \\ A_1 &= y_1 F'(y_0) = 2y_0 y_1 \\ A_2 &= y_2 F'(y_0) + \frac{1}{2!} y_1^2 F''(y_0) = 2y_0 y_1 + y_1^2 \\ &\vdots \end{aligned} \tag{13}$$

Writing the source function in Taylor series form for only 4 terms:

$$g(t) = e^t \approx 1 + t + \frac{t^2}{2} + \frac{t^3}{6} \tag{14}$$

Then we form the recursive relation as in (7):

$$y_0 = f = y(0) + L^{-1}(1 + t + \frac{t^2}{2} + \frac{t^3}{6})$$
$$y_1 = -L^{-1}R(y_0) - L^{-1}(A_0) \quad (15)$$
yields
$$y_{k+1} = -L^{-1}(R(y_k)) - L^{-1}(A_k), \quad k \geq 0$$

$$y_0 = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24}$$
$$y_1 = -\frac{t}{2} - \frac{3}{4}t^2 - \frac{7}{12}t^3 - \frac{5}{16}t^4 - \frac{7}{60}t^5 + \cdots \quad (16)$$
$$y_2 = \frac{3}{8}t^2 + \frac{17}{24}t^3 + \frac{23}{32}t^4 + \cdots$$
$$\vdots$$

After doing much more calculations, we get the solution as:

$$y_T(t) = \sum_{n=0}^{m} y_n = y_0 + y_1 + y_2 + \cdots y_m \quad (17)$$

where $y_T(t)$ denotes the approximate solution computed by using Taylor series expansions. For $m = 4$ we obtain

$$y_T(t) = 1 + 0,5t + 0,125t^2 + 0,020833t^3 + 0,002604t^4 + 0,000520t^5 \quad (18)$$

Now, we make the same calculations by using Chebyshev expansion which can be calculated as in [17,18]:

$$g(t) = e^t \approx \frac{382}{384} + \frac{383}{384}t + \frac{208}{384}t^2 + \frac{68}{384}t^3 + \cdots \quad (19)$$

Again, we have now the recursive relation:

$$y_0 = 1 + 0,9947916t + 0,9973958t^2 + 0,53125t^3 + \cdots$$
$$y_1 = -0.5t - 0,746094t^2 - 1.32857t^3 - 1.19272t^4 + \cdots \quad (20)$$
$$y_2 = -0.125t^2 - 0.207682t^3 - 0.352595t^4 + \cdots$$
$$\vdots$$

By proceeding for $m = 4$ we get

$$y_C(t) = 1 + 0.49947916t + 0.126301t^2 + 0.021991t^3 + 0.002697t^4 + 0.000601t^5 . \quad (21)$$

For larger m more accurate results we get. Figure 1 and Figure 2 displays the errors for only $m = 4$. Table 1 shows the comparisons of these errors for $m = 8$.

Example 2) Consider the Riccati differential equation

$$y' - ty + y^2 = e^{t^2}, \; y(0) = 1 \quad (22)$$

which have the exact solution $y = e^{\frac{t^2}{2}}$.

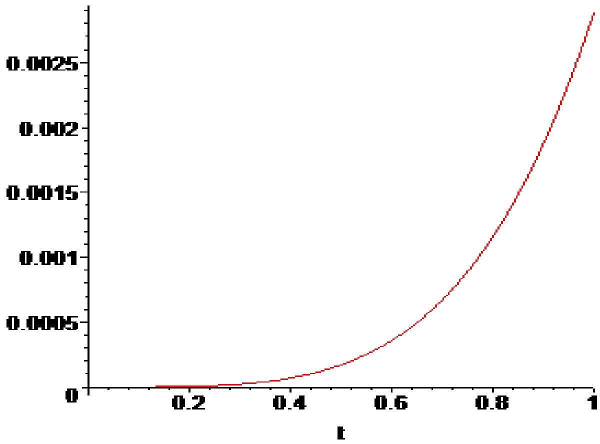

Figure 1. The errors $y - y_T$ for Example 1

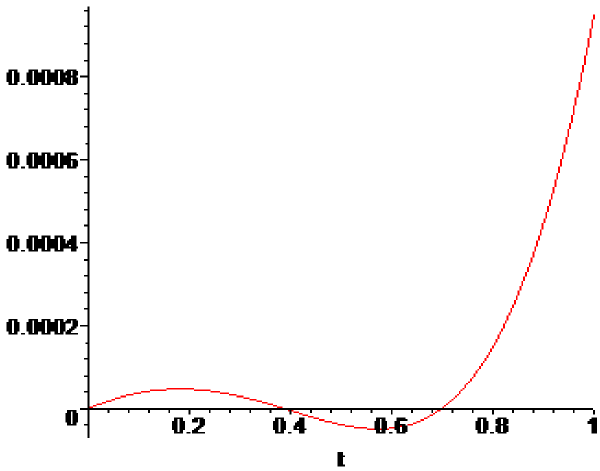

Figure 2. The errors $y - y_C$ for Example 1

Table 1: Absolute errors for $m=8$ for Example 1.

t	$y(t)$	$\lvert y-y_C\rvert$	$\lvert y-y_T\rvert$
0.2	1.1051709	8.054395E-10	6.762634E-8
0.4	1.2214028	5.043533E-8	5.595943E-6
0.6	1.3498588	3.766847E-7	1.939455E-5
0.8	1.4918247	1.504646E-6	4.543433E-4

Solution:
We here have

$$L=\frac{d}{dt}, R(y)=\frac{-t}{2}y, Ny=F(y)=y^2 \text{ and}$$

$$g(t)=e^{t^2}. \quad (23)$$

We can easily compute the Taylor and Chebyshev series expansions of the source terms we need:

$$g_T(t)=e^{t^2}\approx 1+t^2+0.5t^4+0.16666t^6 +0.04166t^8+\cdots \quad (24)$$

and

$$g_C(t)=e^{t^2}\approx 0.99997+1.0001t^2+0.49912t^4 +0.17036t^6+0.034853t^8+\cdots . \quad (25)$$

Following the same procedure as in the previous example 1, we get the approximate solutions:

$$y_T(t)=1+0.5t^2+0.125t^4+0.020833t^6 +0.0026041t^8+0.0002604t^{10}+\cdots \quad (26)$$

and

$$y_C(t)=1+0.49999998t^2+0.12500037t^4+0.02083112t^6 +0.002610400t^8+0.0002514712t^{10}+\cdots$$

Figure 3 and Figure 4 displays the absolute errors for only $m=4$.

We can see the difference even for small m . Table 2 shows the comparisons of these errors for $m=8$.

CONCLUSIONS

In this study, we show that using Chebyshev polynomials is good idea to improve the effectiveness of the Adomian decomposition method.

We use Chebyshev expansions of the source term to obtain more accurate results.

Figures enable us to see that the difference between the using both two methods by graphically.

Tables are also given to show the variation of the absolute errors for larger approximation, namely for larger m .

Maple 18 is used for calculations and sketching graphs.

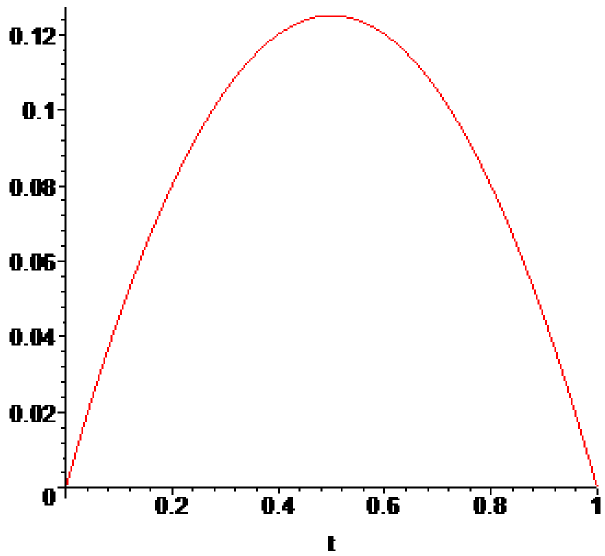

Figure 3: The errors $\lvert y-y_T\rvert$ for Example 2.

Table 2: Absolute errors for $m = 8$ for Example 2.

t	$y(t)$	$\lvert y - y_C \rvert$	$\lvert y - y_T \rvert$
0.2	1.0202013	3.053732E-12	6.039393E-11
0.4	1.0832871	5.958308E-10	1.003932E-10
0.6	1.1972174	1.958503E-9	4.900392E-7
0.8	1.3771278	2.408753E-6	7.540059E-5

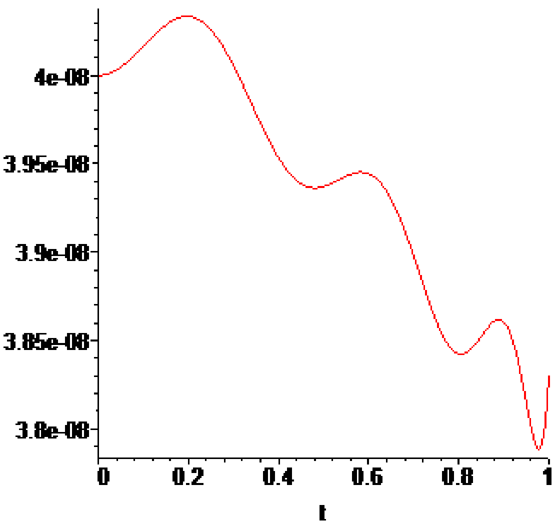

Figure 4: The errors $\lvert y - y_C \rvert$ for Example 2.

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