"HENRI COANDA"
AIR FORCE ACADEMY
ROMANIA

GERMANY

"GENERAL M.R. STEFANIK"
ARMED FORCES ACADEMY
SLOVAK REPUBLIC

INTERNATIONAL CONFERENCE of SCIENTIFIC PAPER
AFASES 2011
Brasov, 26-28 May 2011

ANALYSIS OF MONTE CARLO SIMULATIONS BASED ON DIFFERENT DISCRETIZATION SCHEMAS OF CONTINOUS STOCHASTIC MODELS FROM FINANCE

Tiberiu SOCACIU*, Ioan MAXIM***

*****Faculty of Economics, "Stefan cel Mare" University of Suceava, Romania
** Faculty of Informatics, "Vasile Goldis" West University of Arad, Romania,
*** Department of Teacher Training, "Stefan cel Mare" University of Suceava, Romania

Abstract: *In this paper we will compute with Monte Carlo simulations some derivative values at maturity time in some stochastic models with different discretization schemas (like Euler-Mayurana, Millstein, Runge-Kutta, generic Duffy) and compare results and computing efforts.*

Mathematics Subject Classifications 2010: *65C05, 91G20, 91G60.*
Keywords: *Monte Carlo, financial derivatives, discretization schemas.*

1. BLACK-SCHOLES-MERTON MODEL AND BLACK-SCHOLES PDE

One of popular stochastic equation that models a traded asset (like a stock) is the *Black-Scholes-Merton model* based on *geometric brownian motion* (see [1]):

$$dS(t)=S(t)[\mu dt+\sigma dW(t)] \qquad (1)$$

where $(S(t), t\geq 0)$ is the stochastic process for asset value at timestamp t, $(W(t), t\geq 0)$ is a *Wiener standard process* (see [2]), μ is the *drift rate of return* and σ is *volatility*.

A derivative based on this asset is an other traded asset that fructify at *maturity time* T, payoff depend on value of support, S(T). *payoff* function is defined as:

$$payoff: R_+ \rightarrow R \qquad (2)$$

Main problem is pricing of a financial derivative. For this, we build an risk-free portofolio based on some supports and some derivatives. After applying of Ito lemma (see [3]) we obtain Black-Scholes PDE (see [4]):

$$V_t+½\sigma^2S^2V_{SS}+rSV_S-rV = 0 \qquad (3)$$

for pricing derivatives, where:

$$V: R_+ x[0,T] \rightarrow R_+ \qquad (4)$$

and V(S,t) is value of derivatives at timestamp t if support is valued as S.

Note that for a generalized brownian motion:

$$dS(t)=A(S(t),t)dt+B(S(t),t)dW(t) \qquad (5)$$

where A(S,t) and B(S,t) are some algebraic expression, we can build a generalized form of Black-Scholes PDE (see [5]):

$V_t+½B^2V_{SS}+rSV_S–rV = 0$ (6)

Black-Scholes PDE and generalized Black-Scholes PDE can be linked with Dirichlet condition:

$V(0,t)=0$ (7)
$V(S,T)=payoff(S)$ (8)

that means for 0 value of support, derivatives is valued to 0 too, and value at maturity is payoff function.

2. OTHER MODELS

We give some usual stochastic models in table 1 (see [5]):

Name	**Equations**
Bachelier (see [12])	$dS = adt+bdW$
Black-Scholes-Merton	$dS/S = adt+bdW$
CEV (see [13])	$dS = aSdt+bS^{\beta}dW$
Chen (see [11])	$dS = (\theta(t)–\alpha(t))dt+S^{½}\sigma(t)dW$ $d\alpha = (\zeta(t)–\alpha(t))dt+\alpha(t)^{½}\sigma(t)dW$ $d\sigma = (\beta(t)–\sigma(t))dt+\sigma(t)^{½}\eta(t)dW$
Dias-Rocha (see [6], p. 68)	$dS/S = [k_1(\mu–S)–\lambda k_2]dt+\sigma dW(t)+dq$ $Prob(dq = 0) = 1–\lambda dt$ $Prob(dq = \varphi) = \lambda dt$
Double-Heston (see [7], [8])	$dS = \mu Sdt +v_1^{½}SdW_{11}(t)+v_2^{½}SdW_{12}(t)$ $dv_1 = k_1(\theta_1–v_1)dt+ \xi_1 v_1^{½}dW_1(t)$ $dv_2 = k_2(\theta_2–v_2)dt+ \xi_2 v_2^{½}dW_2(t)$ $dW_1dW_{12} = r_1dt$ $dW_2dW_{22} = r_2dt$
Heston (see [9])	$dS = \mu Sdt+v^{½}SdW_1(t)$ $dv = k(\theta–v)dt+\xi v^{½}dW_2(t)$ $dW_1dW_2 = dt$
Marlim (see [6], p. 68)	$dS = k(\mu–S)dt+\sigma dW(t)+dq$ $Prob(dq = 0) = 1–\lambda dt$ $Prob(dq = \varphi) = \lambda dt$
Merton (diffusion+jumps, see [10], p. 585)	$dS/S = (a–\lambda b)dt+\sigma dW(t)+dq$ $Prob(dq = 0) = 1–\lambda dt$ $Prob(dq = \varphi) = \lambda dt$
SABR (see [14])	$dS = vS^{\beta}dW_1(t)$ $dv = \alpha vdW_2(t)$ $dW_1dW_2 = rdt$
Vasicek	$dS = a(b–S)dt+cdW(t)$

Table 1. Some usual stochastic models.

3. DISCRETIZATION SCHEMAS

For a stochastic process $X(t)_{t\geq 0}$ with next SDE:

$dX(t)=A(X(t),t)dt+B(X(t),t)dW(t)$ (9)

where $W(t)_{t\geq 0}$ is a standard Wiener process, than we can approximate process $X(t)_{t\geq 0}$ with a Markov chain $(Y_n)_{n\geq 0}$, where Y_n is $X(t_n)$. Some usual discretization methods can be found in table 2 (see [5]):

Method name	**Schema**
Explicit Euler-Maruyana (Euler)	$Y_{n+1} = Y_n+A(Y_n,t_n)\Delta+B(Y_n,t_n)N_n\sqrt{\Delta}$

Method name	**Schema**
method (see [15])	
Implicit Euler-Maruyana (Euler) method (see [15])	$Y_{n+1} = Y_n+A(Y_{n+1},t_n)\Delta+B(Y_n,t_n)N_n\sqrt{\Delta}$
θ-stochastic implicit Euler-Maruyana method	$Y_{n+1} = Y_n+A(\theta_n Y_{n+1}+(1-\theta_n)Y_n,t_n+\theta_n\Delta)\Delta+B(Y_n,t_n)N_n\sqrt{\Delta}$
Implicit median point method (see [15])	$Y_{n+1} = Y_n+A((Y_n+Y_{n+1})/2,(t_n+t_{n+1})/2)\Delta+B(Y_n,t_n)N_n\sqrt{\Delta}$
Implicit trapezoidal method (see [15])	$Y_{n+1} = Y_n+(A(Y_{n+1},t_{n+1})+A(Y_n,t_n))\Delta/2+B(Y_n,t_n)N_n\sqrt{\Delta}$
Millstein method (see [16])	$Y_{n+1} = Y_n+A(Y_n,t_n)\Delta+B(Y_n,t_n)\sqrt{\Delta}N_n$ $+\frac{1}{2}B(Y_n,t_n)B_X(Y_n,t_n)\Delta(N_n^2-1)$
A method like Runge-Kutta (see [16])	$Y_{n+1} = Y_n+A(Y_n,t_n)\Delta+B(Y_n,t_n)\sqrt{\Delta}N_n+\frac{1}{2}(B(Y_n+A(Y_n,t_n)\Delta$ $+B(Y_n,t_n)\sqrt{\Delta},t_n)-B(Y_n,t_n)\sqrt{\Delta}(N_n^2-1)$
Generic Duffy method (see [16])	$Y_{n+1} = Y_n+[\alpha C(Y_{n+1},t_{n+1})+(1-\alpha)C(Y_n,t_n)]\Delta$ $+[\eta B(Y_{n+1},t_{n+1})+(1-\eta)B(Y_n,t_n)]\sqrt{\Delta}N_n$ $C(Y,t) = A(Y,t)-\eta B(Y,t)B_X(Y,t)$
Extended Duffy method (see [16])	$Y_{n+1} = Y_n+[\alpha C(Y_{n+1},t_{n+1})+(1-\alpha)C(Y_n,t_n)]\Delta$ $+[\eta B(Y_{n+1},t_{n+1})+(1-\eta)B(Y_n,t_n)]\sqrt{\Delta}N_n$ $+[\xi A(Y_{n+1},t_{n+1})C(Y_{n+1},t_{n+1})+(1+\xi)A(Y_n,t_n)C(Y_n,t_n)]\Delta$ $C(Y,t) = A(Y,t)-\eta B(Y,t)B_X(Y,t)$

Table 2. Some usual discretization schema.

where:

- $\Delta= t_{n+1}-t_n$;
- N_n is ~N(0,1);
- θ_n is a stochastic value in (0,1);
- α is a parameter in (0,1);
- η is a parameter in (0,1);
- ξ is a parameter in (0,1);
- B_X is partial derivative of B on nontemporal dimension.

Note that for a static B (like volatility parameter in Bachelier model) we will have:

$$B_X=0 \qquad (10)$$

Note that for a static A (like drift parameter in Bachelier model) we will have an equivalence between all of explicit and implicit Euler-Maruyana schemas. If A and B are statically, Millstein schema is equivalent too.

If we have a stochastic model with multiple SDEs (like Chen, Heston, double-Heston etc), then we can build a multidimensional Markov Chain: for each SDE we can use any discretization schema like in previous paragraph.

4. NUMERICAL SIMULATIONS

For Bachelier and Black-Scholes-Merton model with:

a = 1
b = 4%
S0 = 9
Payoff(S)=max{0,S-10}
T = 1
Δ =0.01

with Monte Carlo simulation on N=10, 100 and 1000 simulation steps we obtain:

	Pricing value for Bachelier model	Pricing value for Black-Scholes-Merton model	Monte Carlo iterations
Explicit Euler-Maruyana (Euler)	0.00631039	14.1453	10
	0.019516	14.4509	100
	0.0221323	14.5943	1000
Millstein method	0.0287962	14.4454	10
	0.0253927	14.7503	100
	0.0219063	14.6148	1000
Generic Duffy method	0.0168662	13.7366	10
	0.0189815	14.2445	100
	0.0203399	14.2373	1000

Scilab (see [16]) program for this simulation is:

```
function
f=NextValue(CurrentValue,tip,a,b,delta)
N=rand(0,'normal');
radical=sqrt(delta);
patrat=N*N;
// tip>0 Bachelier
// Explicit Euler-Maruyana (Euler)
if tip==1 then
f=CurrentValue+a*delta+b*N*radical;
// Millstein method
elseif tip==2 then
f=CurrentValue+a*delta+b*radical*N;
// A method like Runge-Kutta
elseif tip==3 then
f=CurrentValue+a*delta+b*radical*N+(b-
b*radical*(patrat-1))/2;
// Generic Duffy method
elseif tip==4 then
f=CurrentValue+a*delta+b*radical*N;
// tip<0 Black-Scholes-Merton
// Explicit Euler-Maruyana (Euler)
elseif tip==-1 then
f=CurrentValue*(1+a*delta+b*N*radical);
// Millstein method
elseif tip==-2 then
f=CurrentValue*(1+a*delta+b*radical*N+b*
delta*(patrat-1)/2);
// A method like Runge-Kutta
elseif tip==-3 then
f=CurrentValue*(1+a*delta+b*radical*N+(b
*(1+a*delta+b*radical)-b*radical*(patrat-1)))/2;
// Generic Duffy method Î±=Î·=1/2
elseif tip==-4 then
f=CurrentValue*(1+(a-
b/2)*delta/2+b*radical*N/2)/(-b*radical*N/2-(a-
b/2)*delta/2+1);
end
endfunction

function
f=Path(FirstValue,Maturity,Exercise,tip,a,b,delta)
x=FirstValue;
for t=0:delta:Maturity,
x=NextValue(x,tip,a,b,delta); end;
f=max(0,x-Exercise);
endfunction

function
f=CompleteSimulation(N,FirstValue,Maturity,Exer
cise,tip,a,b,delta)
s=0;
for i=1:N,
s=s+Path(FirstValue,Maturity,Exercise,tip,a,b
,delta);
end;
f=s/N;
endfunction

S0=9;
T=1;
Ex=10;
a=1;
b=0.04;
delta=0.01;
for tip=1:4,
printf("%d %g %g\n",tip, CompleteSimulation(
10, S0, T, Ex, tip, a, b ,
delta), CompleteSimulation( 10, S0, T, Ex, -tip,
a, b , delta));
printf("%d %g %g\n",tip, CompleteSimulation(
100, S0, T, Ex, tip, a, b ,
delta), CompleteSimulation( 100, S0, T, Ex, -
tip, a, b , delta));
```

```
printf("%d %g %g\n",tip, CompleteSimulation(1000, S0, T, Ex, tip, a, b , delta), CompleteSimulation(1000, S0, T, Ex, -tip, a, b , delta));
end;
```

REFERENCES

1. Pavel Fărcaş, Radu Moleriu, *Elemente de probabilități și teoria proceselor stochastice cu aplicații în matematica financiară*, Editura Albastra, Cluj-Napoca, 2006.
2. Daniel Revuz, Marc Yor, *Continuous martingales and Brownian motion*, second edition, Springer-Verlag 1994.
3. Kiyoshi Itō, *On stochastic differential equations*, in *Memoirs*, American Mathematical Society, 1951, 4, pp. 1–51.
4. Fischer Black, Myron Scholes, *The Pricing of Options and Corporate Liabilities*, in *Journal of Political Economy*, 1973, 81 (3), pp. 637–654.
5. Tiberiu Socaciu, *Metode numerice de evaluare a optiunilor financiare*, Editura InfoData, Cluj 2010.
6. Marco Antonio Guimaraes Dias, *Opcoes reais hibridas com aplicacoes empetroleo*, Pontifica Universidade Catolica de Rio de Janeiro, 2005, PhD thesis, online at http://www.puc-rio.br/marco.ind/pdf/tese_doutor_marco_dias.pdf, last access at 2010 mar. 19.
7. Peter F. Christoffersen, Steven L. Heston, Kris Jacobs, *The Shape and Term Structure of the Index Option Smirk: Why Multifactor Stochastic Volatility Models Work so Well*, preprint SSRN, 20 febr. 2009, online at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=961037, last access 2010 mar. 7.
8. Pierre Gauthier, Dylan Possamaï, *Efficient Simulation of the Double Heston model*, preprint SSRN, 2010 jan. 10, online at http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1434853, last access 2010 febr. 12.
9. Steven L. Heston, *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*, in *The Review of Financial Studies*, 1993, Volume 6, number 2, pp. 327–343, online at http://www.javaquant.net/papers/Heston-origina, last access 2010 febr. 12.
10. John C. Hull, *Options, Futures, and other derivatives*, Seventh Edition, Prentice Hall, ISBN 9780135009949, 814 p.
11. Lin Chen, *Stochastic Mean and Stochastic Volatility - A Three-Factor Model of the Term Structure of Interest Rates and Its Application to the Pricing of Interest Rate Derivatives*, Blackwell Publishers, 1996.
12. Louis Bachelier, *Théorie de la spéculation*, in *Annales de l'Ecole Normale Superiure*, Ser. 3, Tome 17, pp. 21-86, 1900, online at http://archive.numdam.org/ARCHIVE/ASENS/ASENS_1900_3_17_/ASENS_1900_3_17__21_0/ASENS_1900_3_17__21_0.pdf, last access 2010 febr. 12.
13. J. C. Cox, *The Constant Elasticity of Variance Option Pricing Model*, in *The Journal of Portfolio Management*, 22, 1996, pp. 15-17.
14. Patrick S. Hagan, Deep Kumar, Andrew S. Lesniewski, Diana E. Woodward, *Managing Smile Risk*, in *Wilmott Magazine*, 2002, pp. 84-108, online at http://www.wilmott.com/pdfs/021118_smile.pdf, last access at 2009 sep. 23.
15. Henri Scurz, *A brief introduction to numerical analysis of (ordinary) stochastic differential equations without tears*, in *IMA Preprint Series* # 1670, December 15,

1999, Institute for Mathematics and its Application, University of Minnesota.
16. Peter E. Kloeden, Eckhard Platen, *Numerical Solution of Stochastic Differential Equations*, Springer, 1999, ISBN 978-3-540-54062-5.
17. *Scilab manual*, online at www.scilab.org/download/5.../manual_scilab-5.2.2_en_US.pdf, last access at 2011 apr. 6.