# SOME RESULTS ON CONVERGENCE IN DISTRIBUTION OF RANDOM VARIABLES

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***Abstract**: This paper presents some results on convergence in distribution of random variables. We define concepts as the weak convergence of probabilities, the function of Kolmogorov, the real Brownian motion, and finally, we refer to the Invariance Principle as one of the notable results in the intervening notions outlined above.*

***Keywords**: convergence in distribution, weak convergence, real Brownian motion.*

## 1. INTRODUCTION

Problems as the convergence's sequence of random variables and the links between certain types of convergence are the subject of a vast literature. It is known that each type of convergence is used in certain situations as well as Brownian motion has a large domain of applicability in the real world.

Starting from these considerations, we stopped in this article on the concepts of weak convergence of probability and convergence in distribution of random variables, notions underpinning the result of a fundamental theory of probability as the Invariance Principle.

## 2. CONVERGENCE IN DISTRIBUTION OF RANDOM VARIABLES

Furthermore, we introduce the function of Kolmogorov (Orman, 2003: 73-74) that uses in the definition of convergence in distribution of random variables.

Let $(\Omega, K, P)$ be -a probability space, F – a set and $f : \Omega \to F$, a function. If we note $K^{(f)}(F)$, the following set:

$$K^{(f)}(F) = \{A \subset F \mid f^{-1}(A) \in K\} \qquad (1)$$

Then, the set $K^{(f)}(F)$ is a new σ-field.

For $(\forall) A \in K^{(f)}(F)$, one can define a single function:

$$P^{(f)}(A) = P\left(f^{-1}(A)\right) \qquad (2)$$

We see now, that the function $P^{(f)}(A)$, defined by that relationship (2) is a probability on σ-field $K^{(f)}(F)$, and thus, the triplet $\left(F, K^{(f)}(F), P^{(f)}\right)$ becomes a probability space.

Indeed, the relationship (2) indicates:

$$P^{(f)}(A) \geq 0, \ (\forall) A \in K^{(f)}(F) \qquad (3)$$

$$P^{(f)}(A) = P\left(f^{-1}(F)\right) = P(\Omega) = 1 \qquad (4)$$

because P is a probability on the σ- field K.

We check now the third axiom of probability, the countable additivity:

If $(A_\alpha)_{\alpha \in I} \subset K^{(f)}(F)$ is a family of events, with $A_{\alpha'} \cap A_{\alpha''} = \emptyset$, if $\alpha' \neq \alpha''$ and $\alpha', \alpha'' \in I$, then:

$$\begin{aligned} P^{(f)}\left(\bigcup_{\alpha \in I} A_\alpha\right) &= P\left(f^{-1}\left(\bigcup_{\alpha \in I} A_\alpha\right)\right) \\ &= P\left(\bigcup_{\alpha \in I} f^{-1}(A_\alpha)\right) = \sum_{\alpha \in I} P\left(f^{-1}(A_\alpha)\right) = \\ &= \sum_{\alpha \in I} P^{(f)}(A_\alpha) \end{aligned} \qquad (5)$$

We used such equality to demonstrate that the relationship $A_{\alpha'} \cap A_{\alpha''} = \emptyset$, for $\alpha' \neq \alpha''$,

$\alpha',\alpha''\in I$ then:

$$f^{-1}(A_{\alpha'})\cap f^{-1}(A_{\alpha''})=f^{-1}(A_{\alpha'}\cap A_{\alpha''})=$$
$$f^{-1}(\varnothing)=\varnothing,$$

that means $f^{-1}(A_{\alpha'})$ and $f^{-1}(A_{\alpha''})$ are incompatible.

From (3), (4) and (5) it results that $\{F, K^{(f)}(F), P^{(f)}\}$ is a probability space.

**Definition 1**

If $(\Omega, K, P)$ is a probability space, the function defined by relationship (2) is called the function of Kolmogorov $\bullet$

Furthermore, we introduce the concepts as "the weak convergence" of probabilities and "the convergence in distribution" of random variables that will be used in two applications (Problem 1 and Problem 2 - Section 3) that we tried to prove (Karatzas, 2005: 60-64).

**Definition 2**

Let $(S,\rho)$ be a metric space, with Borel $\sigma$-field $B(S)$. Let $\{P_n\}_{n\geq 1}$ be a sequence of probabilities on $(S, B(S))$, and let P be another measure on this space.

We say that $\{P_n\}_{n\geq 1}$ *converges weakly to* P and write $P_n \xrightarrow{w} P$, if and only if:

$$\lim_{n\to\infty}\int_S f(s)\,dP_n(s)=\int_S f(s)\,dP(s) \qquad (6)$$

for every bounded, continuous real-valued function f, on S $\bullet$

**Definition3**

Let $\{(\Omega_n, K_n, P_n)\}_{n\geq 1}$ be a sequence of probability spaces and on each of them consider a random variable $X_n$, with values in the metric space $(S,\rho)$. Let $(\Omega, K, P)$ be another probability space, on which a random variable X, with values in $(S,\rho)$ is given.

We say that $\{X_n\}_{n\geq 1}$ *converges to* X *in distribution* and write $X_n \xrightarrow{\mathcal{D}} X$, if the sequence of measures $\{P_n X_n^{-1}\}_{n\geq 1}$ converges weakly to the measure $PX^{-1}$ $\bullet$

It notes that $X_n \xrightarrow{\mathcal{D}} X$ is equivalent with $\lim_{n\to\infty} E_n f(X_n) = Ef(X)$, for every bounded continuous real-valued function f, on S, where E, and $E_n$ denote expectations with respect to $P_n$ and P.

The most important example of distribution convergence is the *Central Limit Theorem*. Lévy formulates and solved the problem: *find the family of all possible limit laws of normed sums of independent and identically distributed random variables*. The answer is given by *The Central Theorem*: "If it exists a series $\{X_n\}_{n\geq 1}$, of independent random variables, identically distributed, with the mean 0 and dispersion $\sigma^2$, then the series:

$$S_n = \frac{1}{\sigma\sqrt{n}}\sum_{k=1}^{n} X_k,\ n\geq 1 \qquad (7)$$

converges in distribution to a normal random variable".

An extension of the Central Limit Theorem is *The Invariance Principle*. This result leads to the property that a sequence of normalized random trajectories will converge in distribution to the Brownian motion.

In the following section, we introduce the definition of the real Brownian motion and *The Invariance Principle*, called The Principle of Donsker (1951) as one of the notable results in the intervening notions outlined above (Karatzas, 2005: 48, 66-70).

## 3. THE INVARIANCE PRINCIPLE

We consider X, a continuous process on a probability space $(\Omega, K, P)$. For $\omega\in\Omega$, the function $t\to X_t(\omega)$ is part of the set $C[0,\infty)$ and it is denoted with $X(\omega)$. We can say in this case, that the random function $X:\Omega\to C[0,\infty)$ is $K\,|\,\beta(C[0,\infty))$ measurable. Thus, if $\{X^{(n)}\}_{n\geq 1}$ is a sequence of continuous processes (each $X^{(n)}$ defined on a separate probability space $(\Omega_n, K_n, P_n)$), the question when $X^{(n)} \xrightarrow{\mathcal{D}} X$ ?

Also, it raises the question of convergence of sequence of finite dimensional distributions $\left\{X^{(n)}\right\}_{n\geq 1}$, to another distribution X, namely, under what conditions the relationship exists:

$$\left(X_{t_1}^{(n)},...,X_{t_d}^{(n)}\right)\xrightarrow{\mathcal{D}}\left(X_{t_1},...,X_{t_d}\right)?$$

**Definition 4**

The standard Brownian motion is a continuous, adapted process $B=\left\{B_t,K_t \mid 0\leq t<\infty\right\}$ where $B_t$ are random variables and $K_t$ - subfield of K defined on the probability space $(\Omega,K,P)$ with the following properties:

i) $B_0=0$;

ii) The increases of $B_t$ are independent, i.e for any finite number of times $0\leq t_1<t_2<...<t_n<\infty$, the random variables $B_{t_2}$-$B_{t_1}$, $B_{t_3}$-$B_{t_2}$,..., $B_{t_n}$-$B_{t_{n-1}}$, are independent;

iii) $(\forall)$ $0\leq s<t<T$, the growth $B_t$-$B_s$ is normally distributed, with mean 0 and dispersion t-s;

iv) $B_t(\omega)$ is a continuous function of t, $(\forall)$ $\omega\in\Omega$ •

We construct now, a sequence of normalized random walks that will converge in distribution to a process that is the Brownian motion.

It considers a sequence of independent random variables $\left\{\xi_j\right\}_{j\geq 1}$ identically distributed, with an average 0 and dispersion $\sigma^2, 0<\sigma^2<\infty$, with the sequence of partial sums $S_0=0$, $S_k=\sum_{j=1}^{k}\xi_k$, $k\geq 1$. With these initial data, we can construct a Brownian motion as following:

We consider now $Y=\left\{Y_t \mid t\geq 0\right\}$, the continuous-time process obtained from the sequence $\left\{S_k\right\}_{k\geq 0}$ by linear interpolation:

$$Y_t=S_{[t]}+\left(t-[t]\right)\xi_{[t]+1},\ t\geq 0 \quad (8)$$

where $[t]\in Z, [t]=\max\left\{k\in Z, k\leq t\right\}$. If we make the graphic in relation both to time and space, we obtain from Y, a sequence of processes $\left\{X^{(n)}\right\}$:

$$X_t^{(n)}=\frac{1}{\sigma\sqrt{n}}Y_{nt},\ t\geq 0 \quad (9)$$

It is noted that if $s=\frac{k}{n}$ and $t=\frac{k+1}{n}$, then the growth $X_t^{(n)}$-$X_s^{(n)}=\frac{1}{\sigma\sqrt{n}}\xi_{k+1}$ is independent of $F_s^{X^{(n)}}=\sigma\left(\xi_1,...,\xi_k\right)$ and in addition, $X_t^{(n)}$- $X_s^{(n)}$ has zero mean and variance t-s. From the previously notes it results that $\left\{X_t^{(n)} \mid t\geq 0\right\}$ is approximately a Brownian motion.

It is known that in the case of the random variables $\xi_j$ are not necessarily normal, but using the Central Limit Theorem it results that the limiting distributions of the increments of $X^{(n)}$ are normal.

**Theorem 1**

If $\left\{X_t^{(n)}\right\}_n$, is the sequence of processes defined by (9) and $0\leq t_1<...<t_d<\infty$, then: $\left(X_{t_1}^{(n)},...,X_{t_d}^{(n)}\right)\xrightarrow{\mathcal{D}}\left(B_{t_1},...,B_{t_d}\right)$, $n\to\infty$, where $\left\{B_t,K_t^B \mid t\geq 0\right\}$, is a standard, one-dimensional Brownian motion•

**Theorem 2** *(The Invariance Principle-Donsker* – 1951*)*

Let $(\Omega,K,P)$ be a probability space, on which we consider, a sequence $\left\{\xi_j\right\}_{j\geq 1}$ of independent, identically distributed random variables with mean zero and finite variance $\sigma^2>0$.

Let us define for every $n\geq 1$, the process $X^{(n)}=\left\{X_t^{(n)} \mid t\geq 0\right\}$, by (9) and this considers $P_n$, the measure induced by $X^{(n)}$ on $\left(C[0,\infty),B\left(C[0,\infty)\right)\right)$. Then $\left\{P_n\right\}_{n\geq 1}$ converges weakly to a measure $P_*$, under which the coordinate mapping process $W_t(\omega)\overset{def}{=}\omega(t)$ on $C[0,\infty)$ is a standard, one-dimensional Brownian motion•

## 4. APPLICATIONS

Furthermore, we solve two problems (Problem 4.5 and Problem 4.12 - Karatzas, 2005: 61, 64). For the second, we give a generalization with $S = C[0,\infty)$.

**Problem 1**

Let us consider a sequence of random variables $\{X_n\}_{n\geq 1}$, with values in the metric space $(S_1,\rho_1)$, which converges in distribution to random variable X. If $(S_2,\rho_2)$ is another metric space and $\varphi : S_1 \to S_2$ continue, then the sequence $\{Y_n\}_{n\geq 1}, Y_n = \varphi(X_n)$ converges in distribution to random variable $Y = \varphi(X)$.

*Demonstration:*

If $(X_n)_n \xrightarrow{\mathcal{D}} X$, then, according to the earlier note:

$$\lim_n E_n f(X_n) = Ef(X) \tag{10}$$

$(\forall) f$ - bounded, continuous and real-valued function $f$ on $S_1$; $S_1$ is a metric space where the random variables $X_n$ take values.

If g is a bounded and real-valued function on metric space $S_2$ and $\varphi : S_1 \to S_2$, then $g \circ f : S_1 \to R$ -bounded, continuous and real-valued function. For $f = g \circ \varphi$, relationship (10) becomes:

$$\lim_n E_n (g\circ\varphi)(X_n) = E(g\circ\varphi)(X) \Leftrightarrow$$

$$\lim_n E_n g(\varphi(X_n)) = Eg(\varphi(X)) \Leftrightarrow$$

$$[\varphi(X_n)]_n \xrightarrow{\mathcal{D}} \varphi(X)$$

that is what needed to be demonstrated •

**Definition 5**

a) Let $(S,\rho)$ be a metric space and $\pi$ a family of probabilities on $(S,B(S))$. We say that $\pi$ is *relatively compact*, if every sequence of elements of $\pi$ contains a weakly convergent subsequence. We say that $\pi$ is *tight* if $(\forall)\varepsilon > 0$ there exists a compact set $K \subseteq S$ such as $P(K) \geq 1-\varepsilon$ $(\forall) P \in \pi$.

b) If $\{X_\alpha\}_{\alpha\in A}$ is a family of random variables, each one defined on a probability space $(\Omega_\alpha, K_\alpha, P_\alpha)$ and taking values in S, we say that this family is relatively compact, if the induced family $\{P_\alpha X_\alpha^{-1}\}_{\alpha\in A}$ has this property •

Furthermore, we propose to explore the convergence of sequence by type:

$$\left(\int_S f_n dP_n\right)_{n\geq 1} \tag{11}$$

where $(P_n)_{n\geq 1}$ is a weak convergent sequence of probabilities defined on Borel $\sigma$-field $B(S)$, S being a metric space. The next enunciation indicates a sufficient condition by convergence of such a full sequence. We consider that the solution of the Problem 2 is standard, but interesting.

**Problem 2**

Let $(S,B(S))$ be a measurable space associated with a metric space $(S,\rho)$, a sequence of probabilities $(P_n)_{n\geq 1}$ and a probability P. Also, let $f_n : S \to R$, $n \geq 1$ a sequence of functions uniformly bounded and converges uniformly on compacts to a continuous function $f$, $f : S \to R$.

If the sequence of probabilities $(P_n)_{n\geq 1}$ converges weakly to P, then $(f_n)_{n\geq 1}$ converges uniformly on compacts to a continuous function f, and if the family of probabilities $\pi = \{P_n, n \geq 1\} \cup \{P\}$ is relatively compact, then the following holds: $\lim_{n\to\infty} \int_S f_n dP_n = \int_S f dP$.

*Demonstration:*

According to the hypothesis of limitation uniform, there exists a constant $a > 0$ such that $|f_n(\omega)| \leq a, (\forall)\omega \in S, (\forall) n \geq 1$ and also:

$$|f(\omega)| \leq a, (\forall)\omega \in S. \tag{12}$$

Let $\varepsilon > 0$ be fixed.

When the family $\pi = \{P_n, n \geq 1\} \cup \{P\}$ is alleged relatively compact, there exists a compact set $K \subset S$, such as:

$$P_n(K) > 1 - \frac{\varepsilon}{6a}, (\forall) n \geq 1. \tag{13}$$

Therefore $P_n(\text{S-K}) < \dfrac{\varepsilon}{6a}, (\forall) n \geq 1.$

The uniformly convergence of $\left(f_n\right)_{n\geq 1}$ to function f on the compact K, ensures the existence of a rank $N_1$, that is natural such that:

$$| f_n(\omega)\text{-}f(\omega)| < \frac{\varepsilon}{3}, (\forall)\, \omega \in K, (\forall)\, n \geq N_1$$

Also, because $P_n \xrightarrow{w} P$, there exists $N_2 \in N^*$ that we have:

$$\left|\int_S f dP_n - \int_S f dP\right| < \frac{\varepsilon}{3}, (\forall)\, n \geq N_2. \qquad (14)$$

We denote $N = \max\{N_1, N_2\}$. For $n \geq N$, we have:

$$\left|\int_S f_n dP_n - \int_S f dP\right| \leq \left|\int_{S\backslash K} (f_n\text{-}f) dP_n\right| + \left|\int_K (f_n\text{-}f) dP_n\right| +$$

$$+\left|\int_S f dP_n \text{-} \int_S f dP\right| \leq \int_{S\backslash K} |f_n\text{-}f| dP_n +$$

$$+\int_K |f_n\text{-}f| dP_n + \left|\int_S f dP_n \text{-} \int_S f dP\right|$$

$$\leq 2aP_n(S\backslash K) + \frac{\varepsilon}{3} P_n(K) + \frac{\varepsilon}{3} < \varepsilon.$$

It results $\int_S f_n dP_n \xrightarrow{n\to\infty} \int_S f dP$ •

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