ON HODRICK-PRESCOTT FILTER. A SHORT SURVEY AND APPLICATIONS

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Abstract: *In this paper, the Hodrick-Prescott filter and some new related results are presented. These results are very useful in macroeconomics and Digital Signal Processing (DSP). Furthermore, an application in the DSP field is given.*

Keywords: *Hodrick-Prescott filter, DSP, macroeconomics.*

1. INTRODUCTION

The Hodrick-Prescott (HP) filter is a tool commonly used in macroeconomics, and it is very usefull to extract a trend component from a time series.In this survey we point out the main results from [1] obtained until now for this type of filter and draw some lines of future research in order to obtain new results. Sakarya and de Jong [1] obtained a new representation of the transformation of the data which is implied by the HP filter. This representation is useful to analyze the properties of the HP filter without relying on ARMA based approximation, which was used in the literature before. Also, we focus on the characterization of the large T behavior of the HP filter and some conditions under where it is asimptotically equivalent to a symmetric weighted average with weights independent of sample size.

Sakarya and de Jong [1] found that the cyclical component of the HP filter also has a weak dependence property when the HP filter is applied to a stationary mixing process, a linear deterministic trend process and/or a process with a unit root. The HP filter is a good tool to achieve weak dependence in time series. One of the techiques that we will pay an important role in our research is the large smoothing parameter approximation of the HP filter, which is derived in [1]. Using this approximation, authors found an alternative justification for the procedure given in 2002 in [2] (for more information, see [1]) for adjusting the smoothing parameter for the data frequency.

2. MAIN RESULTS REGARDING THE HODRICK-PRESCOTT FILTER

2.1 The Hodrick-Prescott filter. The Hodrick-Prescott filter represents a standard tool in macroeconomics, very useful for separating the long trend in data series from short run fluctuations.

The HP filter smoothed series $\hat{\tau}_T = (\hat{\tau}_{T1}, \hat{\tau}_{T2}, \ldots, \hat{\tau}_{TT})$ as defined and described in economics by Hodrick and Prescott [3,4] results from minimizing, over all $\tau \in \mathbf{R}^T$, the function

$$\sum_{t=1}^{T}(y_t - \tau_t)^2 + \lambda \sum_{t=2}^{T-1}(\tau_{t+1} - 2\tau_t + \tau_{t-1})^2, \tag{1}$$

where T denotes the sample size, λ is the nonnegative smoothing parameter so that for quarterly data is often chosen to be equal to 1600, and $y = (y_1, y_2, \ldots, y_t)^t$ represents the data series to be smoothed.

In [5] a similar filtering technique has been introduced, according to [1]. Usually, $\hat{\tau}_{Tt}$ is referred to as the *trend component*, while $\hat{c}_{Tt} = y_t - \hat{\tau}_{Tt}$ is called the *cyclical component*. As stated in [1], there exists a unique minimizer to the minimization problem descried by equation (1), so that, for a known positive defined $(T \times T)$ matrix F_T, by letting I_T denote the $(T \times T)$ identity matrix, $y = (\lambda F_T + I_T)\hat{\tau}_{Tt}$ and

$$\hat{\tau}_{Tt} = (\lambda F_T + I_T)^{-1} y \tag{2}$$

Therefore, the trend component $\hat{\tau}_{Tt}$ and the cyclical component $\hat{c}_{Tt}$ are both weighted averages of y_t and according to [1] we have: $\hat{\tau}_{Tt} = \sum_{s=1}^{T} w_{Tts} y_s$.

For notational convenience, the dependence of w_{Tts} and $\hat{\tau}_{Tt}$ with respect to λ is suppressed. One of the purposes set in [1] is to find a new representation for w_{Tts} and provide immediate consequences of this representation. This approach eliminates the inability to derive a simple analytical formula for the elements of $(\lambda F_T + I_T)^{-1}$, fact that prevented other researchers to find a simple expression for the weights that are implicit for the HP filter (for more information, see [1]).

We shall notice first of all, that: $\hat{\tau}_{Tt}(y_1+1, y_2+1, \ldots, y_T+1) = \hat{\tau}_{Tt}(y_1, y_2, \ldots, y_T) + 1$, so this means that $\sum_{s=1}^{\mathrm{T}} w_{Tts} = 1$ for $t \in \{1,2,\ldots,T\}$. Also, we have that: $\hat{\tau}_{Tt}(1,2,\ldots,T) = t$ and therefore we have that: $\sum_{s=1}^{\mathrm{T}} w_{Tts} s = t$, for $t \in \{1,2,\ldots,T\}$.

Sakarya and de Jong [1] remarked that a quadratic trend is not absorbed in $\hat{\tau}_{Tt}$ in this way. They also noticed that previous literature regarding the HP filter is only based on the observation that the first order condition for $\hat{\tau}_{Tt}$, $t \in \{3,\ldots,T-2\}$, is given by:

$$\begin{aligned} &-2(y_t - \hat{\tau}_{Tt}) - 4\lambda(\hat{\tau}_{T,t+1} - 2\hat{\tau}_{Tt} + \hat{\tau}_{T,t-1}) + 2\lambda(\hat{\tau}_{Tt} - 2\hat{\tau}_{T,t-1} + \hat{\tau}_{T,t-2}) \\ &+ 2\lambda(\hat{\tau}_{T,t+2} - 2\hat{\tau}_{T,t+1} + \hat{\tau}_{Tt}) = 0 \end{aligned} \tag{3}$$

Let $\overline{B}$ denote the forward operator and B the backward operator, then according to [1], this simplifies to the following relation:

$$y_t = (\lambda \overline{B}^2 - 4\lambda \overline{B} + (1+6\lambda) - 4\lambda B + \lambda B^2)\ \hat{\tau}_{Tt}, \tag{4}$$

which can also be re-written as:

$$y_t = (\lambda \,|1-B|^4 + 1)\ \hat{\tau}_{Tt}. \tag{5}$$

Papers that analyze the HP filter based on the first order condition, according to [1], are for example [6-10].

2.2 On the weights of a Hodrick-Prescott filter. In this subsection we shall refer to section 2 from [1] in order to show how the exact weights w_{Ts} implied by the HP filter have been derived. The approach is based on minimizing the function provided in (6) over $(\theta_1, \theta_2, \ldots, \theta_T)$, for a basis of functions $p_j : [0,1] \to \mathbf{R}$, for $t, j \in \mathbf{N}^+$, rather than minimizing the function from (1) over τ :

$$\begin{aligned}
&\sum_{t=1}^{T}\left[y_t - \sum_{j=1}^{T}\theta_j p_j\left(\frac{t}{T}\right)\right]^2 + \lambda\sum_{t=2}^{T-1}\left[\sum_{j=1}^{T}\theta_j p_j\left(\frac{t+1}{T}\right) - 2\sum_{j=1}^{T}\theta_j p_j\left(\frac{t}{T}\right) + \sum_{j=1}^{T}\theta_j p_j\left(\frac{t-1}{T}\right)\right]^2 \\
&= \sum_{t=1}^{T}(y_t - \theta' p_{Tt})^2 + \lambda\sum_{t=2}^{T-1}(\theta' \Delta^2 p_{T,t+1})^2,
\end{aligned} \tag{6}$$

where $\theta' = (\theta_1, \theta_2, \ldots, \theta_T)$ and $p_{Tt}' = (p_1(\tfrac{t}{T}), p_2(\tfrac{t}{T}), \ldots, p_T(\tfrac{t}{T}))$. By differentiating (6) with respect to θ, the minimizer $\hat{\theta}$ derived in [1] is given by:

$$0 = -2\sum_{t=1}^{T} y_t p_{Tt} + 2\sum_{t=1}^{T} p_{Tt} p_{Tt}' \hat{\theta} + 2\lambda\sum_{t=2}^{T-1}(\Delta^2 p_{T,t+1})(\Delta^2 p_{T,t+1})'\hat{\theta}, \tag{7}$$

If an inverse exists, the minimize $\hat{\theta}$ can be expressed as follows:

$$\hat{\theta} = \left(T^{-1}\sum_{t=1}^{T} p_{Tt} p_{Tt}' + \lambda T^{-1}\sum_{t=2}^{T-1}(\Delta^2 p_{T,t+1})(\Delta^2 p_{T,t+1})'\right)^{-1} T^{-1}\sum_{t=1}^{T} y_t p_{Tt}. \tag{8}$$

In [1], authors have chosen $p_1\left(\frac{t}{T}\right) = 1$ and $p_j\left(\frac{t}{T}\right) = \sqrt{2}\cos\left(\frac{\pi(j-1)(t-\frac{1}{2})}{T}\right)$, $j = 2, \ldots, T$.

In what follows, I_T denotes the identity matrix of dimension $(T \times T)$.

Lemma [1]. Let $p_{Tt} = (p_1(\tfrac{t}{T}), p_2(\tfrac{t}{T}), \ldots, p_T(\tfrac{t}{T}))'$, where $p_1\left(\frac{t}{T}\right) = 1$ and

$p_j\left(\frac{t}{T}\right) = \sqrt{2}\cos\left(\frac{\pi(j-1)(t-\frac{1}{2})}{T}\right)$, $j = 2, \ldots, T$. Then, we have:

$$T^{-1}\sum_{t=1}^{T} p_{Tt} p_{Tt}' = I_T \tag{9}$$

and

$$T^{-1}\sum_{t=2}^{T-1}(\Delta^2 p_{T,t+1})(\Delta^2 p_{T,t+1})' = D_T - 32T^{-1}q_{T1}q_{T1}^{'} - 32T^{-1}q_{T2}q_{T2}^{'}, \tag{10}$$

where $D_T = \operatorname{diag}\left(\left\{16\sin\left(\frac{\pi(j-1)}{2T}\right)^4, j=1,\ldots,T\right\}\right)$ and

$q_{T1} = (q_{T11}, q_{T12}, \ldots, q_{T1T})'$, $q_{T2} = (q_{T21}, q_{T22}, \ldots, q_{T2T})'$, where, for $j = 1,2,\ldots,T$,

$q_{T1j} = \sin\left(\frac{\pi(j-1)}{2T}\right)^2 \cos\left(\frac{\pi(j-1)}{2T}\right)$ and $q_{T2j} = \sin\left(\frac{\pi(j-1)}{2T}\right)^2 \cos\left(\frac{\pi(j-1)(T-\frac{1}{2})}{2T}\right)$.

The proof of this result can be found in the Mathematical Appendix of [1].

As authors in [1] point out, the importance of this result that the matrix to be inverted is now 'close' to an easily invertible diagonal matrix (in the way that two matrices of rank 1 have been added to the diagonal matrix).

Remark. The difference from the classical approaches which can be found in literature is the fact that they minimize equation (1) over τ, and no such structure occurs, as the one mentioned above.

It is well known from the literature that explicit formulas can be obtained for the inverse of the sum of a matrix plus another matrix of rank 1, and such results can be adapted to deal with the inverse of a matrix plus a matrix of rank 2 as well. The main purpose of [1] and also of [11] is to use such a result, to obtain a tractable expression for $\hat{\tau}_{Tt}$.

For $m \in \mathbf{Z}$, $\lambda \in [0,\infty)$ and $T \geq 1$ we define:

$$f_{T\lambda}(m) = \frac{1}{2T} + \frac{(-1)^m}{1+16\lambda}(2T)^{-1} + T^{-1}\sum_{j=2}^{T}\cos\left(\frac{\pi(j-1)m}{T}\right)\left(1+16\lambda\sin\left(\frac{\pi(j-1)}{2T}\right)^4\right)^{-1} \tag{11}$$

and for $m \geq 1$, $\lambda \in [0,\infty)$ and $T \geq 1$,

$$\begin{aligned} g_{T\lambda} &= T^{-1}p_{Tm}^{'}(1+\lambda D_T)^{-1}q_{T1} \\ &= T^{-1}\sum_{j=1}^{T}\sqrt{2}\cos\left(\frac{\pi(j-1)(m-\frac{1}{2})}{T}\right)q_{T1j}\left(1+16\lambda\sin\left(\frac{\pi(j-1)}{2T}\right)^4\right)^{-1}. \end{aligned} \tag{12}$$

We also define, similarly to [1], the following sequences:

$$\delta_{T\lambda} = T^{-1}q_{T1}^{'}(I_T + \lambda D_T)^{-1}q_{T1}, \tag{13}$$

$$\eta_{T\lambda} = T^{-1}q_{T1}^{'}(I_T + \lambda D_T)^{-1}q_{T2}, \tag{14}$$

$$\begin{aligned} \xi_{T\lambda} &= 32\lambda(1-64\lambda\delta_{T\lambda})(1-64\lambda\delta_{T\lambda} + 32^2\lambda^2(\delta_{T\lambda}^2 - \eta_{T\lambda}^2))^{-1} \\ &+ 32^2\lambda^2(1-64\lambda\delta_{T\lambda} + 32^2\lambda^2(\delta_{T\lambda}^2 - \eta_{T\lambda}^2))^{-1}\delta_{T\lambda} \end{aligned} \tag{15}$$

and

$$\phi_{T\lambda} = 32^2\lambda^2(1-64\lambda\delta_{T\lambda} + 32^2\lambda^2(\delta_{T\lambda}^2 - \eta_{T\lambda}^2))^{-1}\eta_{T\lambda}. \tag{16}$$

It is clear that: $f_{T\lambda}(m) = f_{T\lambda}(-m), \forall m \in \mathbf{Z}$.
We state now the following result, from [1]:

Theorem [1]. For any $\lambda \in [0,\infty)$ we have $\hat{\tau}_{Tt} = \sum_{s=1}^{T} w_{Ts} y_s$, where:

$$\begin{aligned} w_{Ts} &= f_{T\lambda}(t-s) + f_{T\lambda}(T)I(t+s-1=T) + f_{T\lambda}(t+s-1)I(t+s-1<T) \\ &+ f_{T\lambda}(2T-t-s+1)I(t+s-1>T) + \xi_{T\lambda}g_{T\lambda}(t)g_{T\lambda}(s) + \phi_{T\lambda}g_{T\lambda}(T-t+1)g_{T\lambda}(s) \\ &+ \phi_{T\lambda}g_{T\lambda}(t)\xi_{T\lambda}g_{T\lambda}(T-s+1)g_{T\lambda}(s) + \xi_{T\lambda}g_{T\lambda}(T-t+1)g_{T\lambda}(T-s+1) \\ &= \sum_{j=1}^{T} w_{Tts}^{j}, \end{aligned} \tag{17}$$

with $|f_{T\lambda}(0)| \le 1$ and for any $m \in \{1,2,\dots,T\}$, $|f_{T\lambda}(m)| \le Cm^{-3}$, for some constant C not depending on T .

3. FUTURE WORK ON HP FILTER

Future work will be continued for the mathematical properties of the Hodrick-Prescott filter and find other original calculations of the explicit weights of this filter. We aim to obtain a weak law of large numbers result for functions of the cyclical component.Also, for future research we want to analyze what happens when the sample size and the smoothing parameter are large, independent of what the authors did in [1], and we want to obtain a new procedure for adjusting the smoothing parameter for the data frequency (a similar procedure was obtained also in [2]). We keep in mind that it is possible to utilize the HP filter in combination with other types of filters used in DSP, optimizing the whole filtering process. Also we will review the properties of the HP filter that will helps us in our future work:

- The cut-off region is not steep; this means that leakage from cycles just outside the target region can be significant.
- In engineering applications filter leakage represents a sign of a poor filter.
- Still, in business cycle analysis, researchers have arguments to support at least a small degree of desirable leakage.
- The frequency band of 1.5 to 8 years has been selected based on the expert decision.
- The boundaries 1.5 and 8 years should not be regarded as carved in stone.
- The filter leakage for example will allow strong 9 year cycles to appear in the filtered series.
- The HP filter is asymmetric. Except the central values the double HP filtered series are phase shifted compared to the underlying ideal cycle. Also the phase shifts fade out for a given data as newer data arrive.

CONCLUSIONS

This survey paper points out the new mathematically rigorous results and properties of the HP filter obtained by the authors in [1] and also establishes new lines of research in this field, having set for the future, clear objectives for what kind of results to be obtained. We also discuss about future possible applications of the Hodrick-Prescott filter in DSP, justifying this in section 3.

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