A PRACTICAL APPROACH OF A CERTAIN CLASS OF DYNAMICAL SYSTEMS

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Abstract: *The differential equations and system of differential equations represent the kernel of the mathematical modeling, offering tools to predict the natural phenomenon from science, technics, medicine, biology, etc. In this study we will present the general case of a system of differential equations with many applications is engineering and we will derive all properties of its trajectory. The present study starts from the analyze of a dynamical system which trajectory is an ellipse. It is realized a classification of trajectories. The equation of trajectory is reduced to the canonical form to simplify the calculus. Different geometrical properties of the trajectory are deduced using the analytical and differential study. At the end of this study it is formulated a property regarding the Podar of the trajectory. The trajectory will be represented using the Matlab software.*

Keywords: *dynamical systems, geometrical properties*

1. INTRODUCTION

All mechanism that evolve in time can be represented through a dynamical system. Elementary examples can be found in mechanics, computer science and medicine. The most important thing is the evolution of the system, that is represented by the functions that desctivbe the state of the system as a function of time and satisfy the equation of motion of the system, [1,2,6]. The dynamical systems are encountered also in chemistry. In the paper [3] is studied a chemical phenomenon, an example of an autocatalytic reaction. Using the stability in first approximation and the theory of bifurcations is studied the stability the autocatalytic reaction. The fractals can be also interpreted as dynamical sistems. Its geometry can be seen as a language that describes models and analyzes complex forms from nature. The basics of fractal geometry are algorithms that can be visualized as structures and different forms using the computer, [4,5].

Let us conside the following system of differential equations of first degree:

$$\begin{cases} \dot{x} = ax - by \\ \dot{y} = ax - ay \end{cases} \tag{1}$$

where $ab - a^2 = b^2$, $a, b, k \in \mathrm{Z}$. The characteristic polynomial is:

$$P(\lambda) = det(A - \lambda I_2) = \begin{vmatrix} a-\lambda & -b \\ a & -a-\lambda \end{vmatrix}$$

that leads to the following algebraic equation of second degree:
$\lambda^2 - a^2 + ab = 0 \Leftrightarrow \lambda^2 + k^2 = 0$
with the roots of characteristic equation that are the eigenvalues of the system: $\lambda_{1,2} = \pm ki$.
The general solution of the above system is:

$$x(t) = C_1 \cos kt + C_2 \sin kt$$
$$y(t) = \frac{aC_1 - kC_2}{b} \cos kt + \frac{aC_2 + kC_1}{b} \sin kt \qquad (2)$$

Taking into account the initial conditions:
$x(0) = x_0; y(0) = y_0$
we obtain the values of the constants from te general solution: $C_1 = x_0; C_2 = \frac{ax_0 - by_0}{k}$
therefore the solution of the above system with initial condition is:

$$x(t) = x_0 \cos kt + \frac{ax_0 - by_0}{k} \sin kt$$
$$y(t) = y_0 \cos kt + \frac{ab(x_0 - y_0)}{k} \sin kt \qquad (3)$$

Next we want to prove that the trajectory of the system (1) is an ellipse. Deriving of the trajectory equation is made by solving the differential equation total exact obtained from the system in the following way:
$$\frac{\dot{y}}{\dot{x}} = \frac{ax - ay}{ax - by} \Leftrightarrow (ax - ay)dx - (ax - by)dy = 0$$

It is necessary to find a function $F(x, y) = C$ whose differentiate of first degree is:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = (ax - ay)dx - (ax - by)dy = 0$$

Through identification we have to solve the following system:

$$\frac{\partial F}{\partial x} = ax - ay$$

$$\frac{\partial F}{\partial x} = -ax + by$$

and the solution of the differential total exact equation will be:
$$F(x, y) = \frac{ax^2}{2} - axy + \frac{by^2}{2} = C$$

By computation we obtain the trajectory equation:
$$(ax - y)^2 + (b - 1)y^2 = C \qquad (4)$$
that represents the equation of an ellipse.

2. THE CLASSIFICATION OF TRAJECTORY

In this section we wand to realize a classification of the conics from this family. We consider our particular conic:

$(\Gamma): a^2x^2 - 2axy + by^2 - c = 0$

and by comparing with the general form of a conic:

$(\Gamma): a_{11}x^2 + 2a_{12}xy + a_{22}y^2 + 2a_{13}x + 2a_{23}y + a_{33} = 0$

we make the identifications: $a_{11} = a^2, a_{12} = -a, a_{22} = b, a_{13} = a_{23} = 0, a_{33} = -c$.

Therefore the big invariant of the conic will be:

$$\Delta = \begin{vmatrix} a^2 & -a & 0 \\ -a & b & 0 \\ 0 & 0 & -c \end{vmatrix} = -a^2c(b-1)$$

For the case when $a \neq 0, c \neq 0$ and $b \neq 1$ the conic (Γ) is non degenerate.

The small invariant of the conic is:

$$\delta = \begin{vmatrix} a^2 & -a \\ -a & b \end{vmatrix} = a^2(b-1)$$

Next we have to discuss some cases:

1. If $\delta = 0$ then we obtain that $\Delta = 0$. In this situation the conic (Γ) can not be a parabola $\forall a,b,c \in \mathbb{R}$. If $a = 0$ or $b = 1$ then the big invariant in also null: $\Delta = 0$ and we obtain that the conic could be formed by two lines: $(\Gamma) = (d_1) \cup (d_2)$ where $(d_1) \parallel (d_2)$ or $(d_1) = (d_2)$, or the conic is empty: $(\Gamma) = \emptyset$
2. If the small invariant is negative: $\delta < 0$ then we have $b < 1, a, c \neq 0$ and the conic (Γ) is a hyperbola.
3. If the small invariant is positive $\delta > 0$ and $I \cdot \Delta < 0$ with $a, c \neq 0, b \neq 0$ then the conic (Γ) is an ellipse. Here: $I = a_{11} + a_{22} = a^2 + b$. The above conditions give us the following restrictions for the elliptic case: $c > 0, b > 1, a \neq 0$.

Taking into account that the trajectory of our system (1) is an ellipse we will consider the third case into a particular representation. We chose: $a = 1, b = 2, c = 1$ then the conic is: $(\Gamma): x^2 - 2xy + 2y^2 - 1 = 0$

3. THE CANONICAL FORM OF THE TRAJECTORY

Because $a_{12} = -1 \neq 0$ we will realize a rotation of the axis. Let A be the matrix of the quadratic form of conic's equation:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

Then the characteristic equation corresponding to the matrix A is: $det(A-\lambda I_2)=0$ that leads us to the following algebraic equation: $\lambda^2-3\lambda+1=0$ having the roots:

$$\lambda_{1,2}=\frac{3\pm\sqrt{5}}{2}.$$

The canonical form of the quadratic form is: $\lambda_1 x'^2+\lambda_2 y'^2$. We do not effectuate the translation in the cnter because the general equation of the conic (Γ) does not contain terms of first degree. We chose λ_1 and λ_2 i.e. $sgn(\lambda_1-\lambda_2)=sgn(a_{12})$ and we will obtain:

$\lambda_1=\frac{3-\sqrt{5}}{2},\lambda_1=\frac{3+\sqrt{5}}{2}$. The canonical form of the ellipse equation is:

$$(\Gamma):\frac{x'^2}{\frac{3+\sqrt{5}}{2}}+\frac{y'^2}{\frac{3-\sqrt{5}}{2}}-1=0$$

Applying the roto-translation method we obtain the angle θ with which the reference coordinate rotates: $tg2\theta=\frac{2a_{12}}{a_{11}-a_{22}}\Leftrightarrow tg^2\theta+tg\theta-1=0\Leftrightarrow tg\theta_{1,2}=\frac{-1\pm\sqrt{5}}{2}$.

Because $tg2\theta=2>0\Leftrightarrow 2\theta<\pi\Leftrightarrow\theta<\frac{\pi}{2}\Leftrightarrow tg\theta>0\Leftrightarrow tg\theta=\frac{-1+\sqrt{5}}{2}$. Taking into account that $\cos^2\theta=\frac{1}{1+tg^2\theta}$ using basic computations we will obtain that: $\frac{2}{\sqrt{10-2\sqrt{5}}}>0$ and $\sin\theta=\frac{\sqrt{10-2\sqrt{5}}}{2\sqrt{5}}>0$.

The rotation matrix for the base change is:

$$S=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}\Leftrightarrow S=\begin{pmatrix}\frac{2}{\sqrt{10-2\sqrt{5}}} & -\frac{\sqrt{10-2\sqrt{5}}}{2\sqrt{5}}\\ \frac{\sqrt{10-2\sqrt{5}}}{2\sqrt{5}} & \frac{2}{\sqrt{10-2\sqrt{5}}}\end{pmatrix}.$$

From $X=SX'$ we obtain the connection between the coordinates before the rotation and the coordinates after the rotation.

We want to determine the center of symmetry of ellipse in the initial reference system. Considering the function: $f(x,y)=x^2-2xy+2y^2-1$ we will solve the below system of equations:

$$C:\begin{cases}\frac{\partial f}{\partial x}(x,y)=0\\ \frac{\partial f}{\partial y}(x,y)=0\end{cases}\Leftrightarrow\begin{cases}2x-2y=0\\ -2x+4y=0\end{cases}$$

and the solution is $O(0,0)$ that expresses the missing of the translation.

The differential study on the conic is realized in the reference coordinates system: $(x'Oy')$:

$$(\Gamma): \frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} - 1 = 0$$

where the parametrisation of the ellipse is:

$$(\Gamma): \begin{cases} x' = a'\cos t \\ y' = b'\sin t \end{cases} \quad \text{where} \quad a' = \sqrt{\frac{3+\sqrt{5}}{2}}, b' = \sqrt{\frac{3-\sqrt{5}}{2}}$$

The equation of the osculator circle for the elliptical trajectory in $A(t_0 = 0) \Leftrightarrow A(a',0)$ is: $(\mathrm{C}):(X-\alpha)^2 + (Y-\beta^2) - R^2 = 0$,

where $\alpha = x_0' - \frac{\dot{y}_0'(\dot{x}_0'^2 + \dot{y}_0'^2)}{\dot{x}_0'\ddot{y}_0' - \ddot{x}_0'\dot{y}_0'} = \frac{3\sqrt{2}\sqrt{3-\sqrt{5}}}{2}$, $\beta = y_0' + \frac{\dot{x}_0'(\dot{x}_0'^2 + \dot{y}_0'^2)}{\dot{x}_0'\ddot{y}_0' - \ddot{x}_0'\dot{y}_0'} = 0$ and $R = \frac{(\dot{x}_0'^2 + \dot{y}_0'^2)^{\frac{3}{2}}}{|\dot{x}_0'\ddot{y}_0' - \ddot{x}_0'\dot{y}_0'|} = \frac{3-\sqrt{5}}{2}$.

The equation of the osculator circle for the ellipse in $A(a',0)$ is:

$$(\mathrm{C}): \left(X - \frac{3\sqrt{2}\sqrt{3-\sqrt{5}}}{2} \right)^2 + Y^2 - \left(\frac{3-\sqrt{5}}{2} \right)^2 = 0$$

The curvature of the elliptical trajectory is:

$$K(t) = \frac{\dot{x}'(t)\ddot{y}'(t) - \ddot{x}'(t)\dot{y}'(t)}{(\dot{x}'^2(t) + \dot{y}'^2(t))^{\frac{3}{2}}} = \frac{ab}{a'^2\sin^2 t + b'^2\cos^2 t} > 0, \forall t \in \mathrm{R}$$

The length of the curve bow $\mathrm{L}_{A_1B_1}$ between two points $A(t_{A_1})$ and $B(t_{B_1})$ found on the elliptical trajectory is:

$$\mathrm{L}_{A_1B_1} = \left| \int_{t_{A_1}}^{t_{B_1}} \sqrt{\dot{x}'^2(t) + \dot{y}'^2(t)}\,dt \right| = \left| \int_{t_{A_1}}^{t_{B_1}} \sqrt{a'^2 + \cos^2(b'^2 - a'^2)}\,dt \right|$$

Next we compute the tangent and the normal in a point $M(t)$ at the elliptical trajectory

$$(\Gamma):\begin{cases} x' = a'\cos t \\ y' = b'\sin t \end{cases} :$$

$$\left(d_{tg|M}\right): Y - y'(t) = \frac{\dot{y}'(t)}{\dot{x}'(t)}(X - x'(t)) \Leftrightarrow$$

$$\left(d_{tg|M}\right): Y - \sqrt{\frac{3-\sqrt{5}}{2}}\sin t = \frac{\sqrt{5}-3}{2}ctgt\left(X - \sqrt{\frac{3+\sqrt{5}}{2}}\cos t\right)$$

$$\left(d_{n|M}\right): Y - y'(t) = -\frac{\dot{x}'(t)}{\dot{y}'(t)}(X - x'(t)) \Leftrightarrow$$

$$\left(d_{n|M}\right): Y - \sqrt{\frac{3-\sqrt{5}}{2}}\sin t = \frac{2}{\sqrt{5}-3}tgt\left(X - \sqrt{\frac{3+\sqrt{5}}{2}}\cos t\right)$$

The segment of the tangent, the segment of the normal, the sub-tangent and the subnormal of the elliptical trajectory $(\Gamma): \frac{x'^2}{a'^2} + \frac{y'^2}{b'^2} - 1 = 0$, where $a' = \sqrt{\frac{3+\sqrt{5}}{2}}$ and $b' = \sqrt{\frac{3-\sqrt{5}}{2}}$ in the regular point $M(x', y')$ will be derived in the following. The lengths of the forth segment are computing using the formula:

$$S_{tg} = \left|\frac{y'}{m}\right|\sqrt{1+m^2}, S_n = |y'|\sqrt{1+m^2}, S_{stg} = \left|\frac{y'}{m}\right|, S_{sn} = |y'm|,$$

where $m = -\frac{dy}{dx} = -\frac{F_x^{'}}{F_y^{'}} = -\frac{b^{'2}}{a^{'2}}\frac{x'}{y'}$ and $y' = f(x)$ is the explicit representation of the curve: $(\Gamma): y' = \pm\frac{b'}{a'}\sqrt{a^{'2} - x^{'2}}$.

Replacing in the above formula we obtain the value of $m = \mp\frac{b'}{a'}\frac{x'}{sqrta^{'2} - x^{'2}}$ and the lengths of the found elements will be:

$$S_{tg} = \left|\frac{a^{'2} - x^{'2}}{x'}\right|\sqrt{1+\left(\frac{b'}{x'}\right)^2 \frac{x^{'2}}{a^{'2} - x^{'2}}}, S_n = \left|\frac{b'}{a'}\sqrt{a^{'2} - x^{'2}}\right|\sqrt{1+\left(\frac{b'}{x'}\right)^2 \frac{x^{'2}}{a^{'2} - x^{'2}}}$$

$$S_{stg} = \left|\frac{a^{'2} - x^{'2}}{x'}\right|, S_{sn} = \left|\left(\frac{b'}{a'}\right)^2 \cdot x'\right|.$$

Proposition 1 *The geometrical place of the projections of a fixed point* I *on the tangents at the elliptical trajectory* $(Gamma)$ *is the Booth leminscate. (the podar trajectory)*

Prove. We consider the vectorial equation of the curve $(\Gamma):\vec{r}=\vec{r}(t)$, where $\vec{r}(t)=a'\cos t\vec{i}+b'\sin t\vec{j}$ and $I(x_0,y_0)$ is a fixed point having the position vector $\vec{r}_0$. The position vector $\vec{R}$ of the projection P of the point I on the tangent vector $\dot{\vec{r}}$ in a regular point M with the position vector $\vec{r}$ of the curve (Γ) is $\vec{R}=\vec{r}+\lambda\dot{\vec{r}}$. To find the scalar λ we will use the orthogonality of the vectors $\dot{\vec{r}}$ and $\overleftarrow{IP}$ where:

$$\overleftarrow{IP}=\vec{R}-\vec{r}_0=\vec{r}+\lambda\dot{\vec{r}}-\vec{r}_0$$

Multiplying the above equality with $\dot{\vec{r}}$ we obtain:

$$0=\vec{r}\cdot\dot{\vec{r}}+\lambda\|\dot{\vec{r}}\|-\vec{r}_0\cdot\dot{\vec{r}}\Leftrightarrow\lambda=-\frac{(\vec{r}-\vec{r}_0)\cdot\dot{\vec{r}}}{\|\dot{\vec{r}}\|}$$

The vectorial equation of the podar trajectory is:

$$(\mathrm{P}):\vec{R}=\vec{r}-\frac{(\vec{r}-\vec{r}_0)\cdot\dot{\vec{r}}}{\|\dot{\vec{r}}\|}\cdot\dot{\vec{r}}$$

and the parametric equations of the podar are:

$$(\mathrm{P}):\begin{cases}X=x'-\dfrac{\dot{x}'(x'-x_0)+\dot{y}'(y'-y_0)}{\dot{x}^2+\dot{y}^2}\dot{x}'\\ Y=y'-\dfrac{\dot{x}'(x'-x_0)+\dot{y}'(y'-y_0)}{\dot{x}^2+\dot{y}^2}\dot{y}'\end{cases}$$

In our case the parametric equations of the Podar of elliptic trajectory

$$(\Gamma):\begin{cases}x'=a'\cos t\\ y'=b'\sin t\end{cases}$$ in report with $I=O,x_0=y_0=0$ are:

$$(\mathrm{P}):\begin{cases}X=\dfrac{a'b'^2\cos t}{a'^2\sin^2 t+b'^2\cos^2 t}\\ Y=\dfrac{a'^2b'\sin t}{a'^2\sin^2 t+b'^2\cos^2 t}\end{cases}$$

from the above equations through elimination of t the implicit equation of the searched podar is the Booth lemniscate:

$$(\mathrm{P}):a'^2X^2+b'^2Y^2-(X^2+T^2)^2=0$$

where $a'=\sqrt{\dfrac{3+\sqrt{5}}{2}},b'=\sqrt{\dfrac{3-\sqrt{5}}{2}}$, see Fig. 1

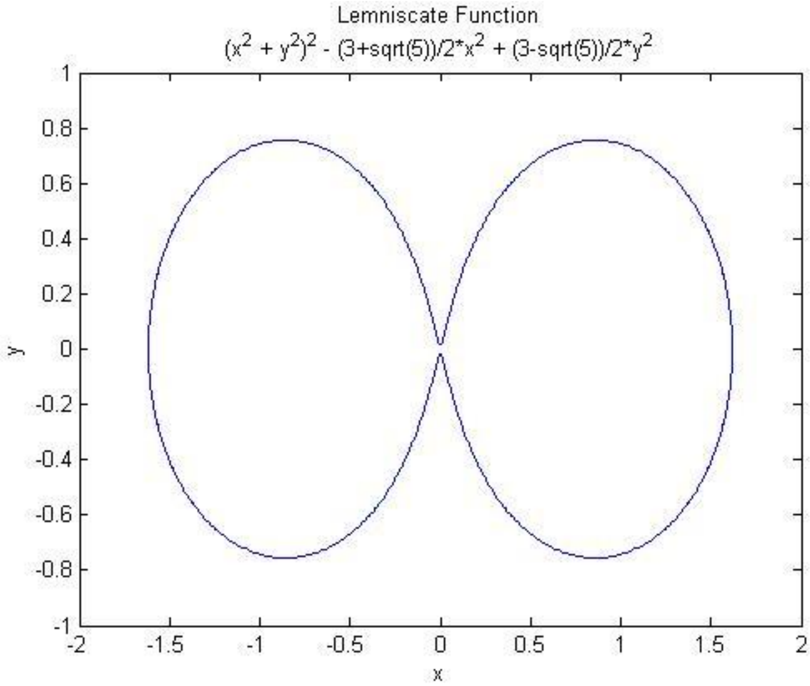

FIG. 1. The Lemniscate function

REFERENCES

[1] N. Lungu and A. Chisalita, *Sisteme dinamice si haos*, MatrixRom, Bucharest, 2007;

[2] A. Bîrzu, G. Bourceanu and L. Onel, *Dinamică neliniară*, MatrixRom, Bucharest, 2003;

[3] O. Florea and M. Purcaru, *Mathematical Modeling And The Stability Study Of Some Chemical Phenomena*, Proceedings of AFASES 2012;

[4] C. Carstea, N. Enache-David and L. Sangeorzan, Fractal model for simulation and inflation control, *Bulletin of the "Transilvania" University of Brasov*, vol 7(56), No. 2, Series III: Mathematics, Informatics, Physics, pp. 161-168, 2014;

[5] N. Enache-David and L. Sangeorzan, *An Application on Web Path Personalization*, Proceedings of the 27th International Business Information Management Association Conference - Innovation Management and Education Excellence Vision 2020:From Regional Development Sustainability to Global Economic Growth, May 4-5, 2016, Milan, Italy, pp. 2843-2848;

[6] L. Sangeorzan and N. David, *Some methods of generating fractals and encoding images*, in Orman,, Proceedings of the 2nd international conference on symmetry and antisymmetry in mathematics, formal languages and computer science, satellite conference of 3ECM, Brasov, Romania, June 29-July 1, 2000. Brasov: Editura Universitatii Transilvania. 337-342

[7] H. Broer and F. Takens, *Dynamical systems and chaos*, Springer, 2009;

[8] C. Rotaru, R. I. Edu, M. Andres-Mihăilă, P. G. Matei, Applications Of Multivariable Control Techniques To Aircraft Gas Turbine Engines, *Review of the Air Force Academy* Vol 12, No 2(26), pp 45-50, 2014;

[9] M.A.P. Purcaru and G. Campean, New Classes of R -Complex Hermitian Finsler Spaces With (α,β)-Metrics, *Review of the Air Force Academy*, Vol 13, No 1(28), pp 97- 100, 2015;

[10] A. Tudorache and I. Fuiorea, Considerations Regarding the Dynamic Balancing of Complex Rotors, *Review of the Air Force Academy*, Vol 13, No 3(30), pp51-54, 2015.