ANALYSIS OF THE FUNDAMENTAL LIMIT PROBLEMS IN PROBABILITY THEORY

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Abstract: *This paper is aimed to emphasize that the DeMoivre-Laplace integral theorem, the Lindeberg-Feller theorem, the Lyapunov's theorem and some others are a good basis for a large variety of problems of fundamental importance to the theory of probability itself and to its multiplicity of applications in the economic sciences, technology, natural sciences, even in the process of transmission of information or in computer science and of course in many others. With the multicriterial analysis method we want to establish which one has the sustainability more efficient.*

Keywords: *approximation, limit theorems, multicriterial analysis , precision, probability*

1. INTRODUCTION

Let v_n be number of successes in n Bernoulli samples. It is assumed that each success has probability p. Then $b(k;n.p)$ is the probability of event $v_n = k$.

Usually we are interested in the probability of the next event: *the number of all the successes that lie between two limits initial data, α and β* . For α and β integers, with $\alpha < \beta$, then this event is defined by the relationship $\alpha \le v_n \le \beta$. Its corresponding probability is given by

$$P[\alpha \le v_n \le \beta] == b(\alpha;n,p) + b(\alpha+1;n,p) + \cdots + b(\beta;n,p). \quad (1)$$

Since the above sum can have many terms, a direct evaluation is impossible.

First DeMoivre and then Laplace realized that whenever n is larger, it can successfully used the normal distribution function, in order to obtain simple approximation of the probability (1).

This is very important as we will see below, not only for numerical computation.

A basic problem is to determine a scheme of independent trials which consist in determining the probability $b(k;n,p)$ that in n trials an event A will occur k times, and that in the rest $n-k$ samples the complementary event $\bar{A}$ will occur.

First we want to find the probability that event $A^{(r)}$ will occur in k specific samples (for example in trials with the numbers $r_1, r_2, \dots, r_k$) and do not occur in the rest $n-k$ samples. But this probability is $p^k q^{n-k}$ (according to the multiplication theorem of independent events). Now, according to the additivity theorem of probability $b(k;n,p)$ is equal to the sum of the probabilities above calculated for all different modes k of occurrences of the event and $n-k$ nonoccurences from among n samples. From combinatorial theory we know that the number of such ways is

$$C_n^k = \binom{n}{k} = \frac{n!}{k!\,(n-k)!}, 0 \le k \le n.$$

Therefore, we obtain for the probability $b(k;n,p)$ the following estimation

$$b(k;n,p) = \binom{n}{k} p^k q^{n-k} = \frac{n!}{k!\,(n-k)!} p^k q^{n-k}. \quad (2)$$

It is noted that for large values of n and k, the computation of probability $b\ (k;\ n,p)$ using the formula (2), involves great difficulties. Thus there is a need to obtain asymptotic formula that allows the calculation of these probabilities with a sufficient degree of accuracy.

Thus, the main step is to obtain an asymptotic formula for (2).

DeMoivre is the first which determined in 1730, such a formula for asymptotic Bernoulli's scheme when $p = q = \frac{1}{2}$. Later this result was generalized by Laplace in arbitrary case $0 < p < 1$.

2. LIMIT THEOREMS

In this way, we have the following limit theorem:

Theorem [(1)] (the DeMoivre-Laplace local limit theorem) *If the probability of occurrence of some event A in n independent trials is constant and is equal to p, $(0 < p < 1)$, and $q = 1 - p$, then, the probability $b(k; n, p)$ that in each of the trials event A will occur exactly k times satisfies the relation*

$$b(k; n, p) : \frac{1}{\sqrt{2\pi npq}} e^{-\frac{x^2}{2}} \to 1 \qquad (3)$$

as $n \to \infty$, uniformly in all k, for which $x_{n,k}$ lies in some finite interval, and verifies the equality

$$x_{n,k} = \frac{k - np}{\sqrt{npq}}, \quad 0 \le k \le n \qquad (4)$$

If M is an arbitrary positive constant set, then for those k for which

$$|x_{n,k}| \le M$$

we have

$$C_n^k p^k q^{n-k} \sim \frac{1}{\sqrt{2\pi npq}} e^{-\frac{x_{n,k}^2}{2}} \qquad (5)$$

(The convergence is relative to n and is uniformly relative to k)

Demonstration

From

$$x_{n,k} = \frac{k - np}{\sqrt{npq}}$$

it results that

$$k = np + \sqrt{npq}\, x_{n,k}$$

$$n - k = nq - \sqrt{npq}\, x_{n,k}.$$

Because $|x_{n,k}| \le M$ we have

$$\begin{cases} \dfrac{k}{np} = 1 + \dfrac{\sqrt{npq}}{np} \cdot x_{n,k} \to 1 \text{ şi deci } k \sim np \\ \dfrac{n-k}{nq} = 1 - \dfrac{\sqrt{npq}}{nq} x_{n,k} \to 1 \text{ şi deci } n - k \sim nq \end{cases}$$

(6)

Using the Stirling formula we can writte

$$C_n^k p^k q^{n-k} \sim$$

$$\sim \frac{\left(\frac{n}{e}\right)^n \sqrt{n}\sqrt{2\pi}}{\left(\frac{k}{e}\right)^k \sqrt{k}\sqrt{2\pi}\left(\frac{n-k}{e}\right)^{n-k}\sqrt{n-k}\sqrt{2\pi}} \cdot p^k q^{n-k} \sim \sqrt{\frac{n}{k(n-k)}} \frac{1}{\sqrt{2\pi}} \varphi(n,k)$$

where

$$\varphi(n,k) = \frac{n^n}{k^n (n-k)^{n-k}} p^k q^{n-k} = \left(\frac{np}{k}\right)^k \left(\frac{nq}{n-k}\right)^{n-k}.$$

Because $k \sim np$ and $n - k \sim nq$ it results that

$$C_n^k p^k q^{n-k} \sim \frac{1}{\sqrt{2\pi npq}} \varphi(n,k).$$

Further we demonstrate that

$$\varphi(n,k) \sim e^{-\frac{x_{n,k}^2}{2}}$$

We use Taylor's expansion of $ln(1+x)$

$$ln(1+x) = x - \frac{x^2}{2} + \cdots + (-1)^{n-1}\frac{x^n}{n} + \cdots for\ |x| < 1$$

and we get

$$\ln\left(\frac{np}{k}\right)^k = k ln\left(\frac{np}{k}\right) \sim k ln\left(\frac{k - \sqrt{npq}x_{n,k}}{k}\right) = k ln\left(1 - \frac{\sqrt{npq}x_{n,k}}{k}\right) =$$

$$= k\left(-\frac{\sqrt{npq}}{k} x_{n,k} - \frac{npq}{2k^2} x_{n,k}^2 - \cdots\right)$$

$$ln\left(\frac{nq}{n-k}\right)^{n-k} = (n-k) ln\left(\frac{nq}{n-k}\right) \sim (n-k) ln\left(\frac{n-k+\sqrt{npq}x_{n,k}}{n-k}\right) =$$

$$= (n-k) ln\left(1 + \frac{\sqrt{npq}x_{n,k}}{n-k}\right) == (n-k)\left(\frac{\sqrt{npq}}{n-k} x_{n,k} - \frac{npq}{2(n-k)^2} x_{n,k}^2 + \cdots\right)$$

because $\left|\frac{\sqrt{npq}}{k} x_{n,k}\right| < 1$ and

$$\left|\frac{\sqrt{npq}}{n-k} x_{n,k}\right| < 1 \ (*)$$

are satisfied for n sufficient large for

$|x_{n,k}| < M$. So

$$ln\varphi(n,k) == ln\left(\frac{np}{k}\right)^k + ln\left(\frac{nq}{n-q}\right)^{n-k} \sim k\left(-\frac{\sqrt{npq}}{k} x_{n,k} - \frac{npq}{2k^2} x_{n,k}^2 - \cdots\right) +$$

$$+(n-k)\left(\frac{\sqrt{npq}}{n-k}x_{n,k}-\frac{npq}{2(n-k)^2}x_{n,k}^2+\cdots\right)=$$

$$=\left(-\sqrt{npq}x_{n,k}-\frac{npq}{2k}x_{n,k}^2-\cdots\right)+\left(\sqrt{npq}x_{n,k}-\frac{npq}{2(n-k)}x_{n,k}^2+\cdots\right)=$$

$$=\frac{npq}{2}x_{n,k}^2\left(-\frac{1}{k}-\frac{1}{n-k}\right)+\cdots=-\frac{n^2pq}{2k(n-k)}x_{n,k}^2+\cdots.$$

We justify why we neglegt the terms of developing the terms with higher level that two for $n\to\infty$. When n is sufficient large, the two quantities from (4) are smaller than $\frac{2}{3}$ and

$$k\left|\frac{\sqrt{npq}}{k}x_{n,k}\right|^3=(n-k)\left|\frac{\sqrt{npq}}{(n-k)}x_{n,k}\right|^3$$

Because $pq<1$ and $|x_k|\le M$it does not exceed

$$\frac{n^{\frac{2}{3}}}{k^2}M^3+\frac{n^{\frac{3}{2}}}{(n-k)^2}M^3$$

which evidently tends to 0 when $n\to\infty$ due to relations (6). Thus, using these relationes, it results that

$$\ln\varphi(n,k)\sim\frac{n^2pq}{2npnq}x_{n,k}=-\frac{x_{n,k}^2}{2}$$

This is equivalent with

$$\varphi(n,k)\sim e^{-\frac{x_{n,k}^2}{2}}$$

And it follows (5).

Remark. The approximation is used if n is sufficient large such that $np\ge 5$ and $n(1-p)\ge 5$.

Some problems require the study of probability theory amounts to a large number of random variables. Central limit theorem establishes the conditions under which the limit distribution of the considered sums is normal.

Theorem [(2)] (DeMoivre-Laplace) *Let A be an eventwhich has the probability of realisations $p=P(A)$ in to an independent trials array. If S_n is the number of realisations of a in n trials, then for any a and b, $a<b$,*

$$\lim_{n\to\infty}P\left(\left\{a\le\frac{S_n-np}{\sqrt{npq}}\le b\right\}\right)==\frac{1}{\sqrt{2\pi}}\int_a^b e^{-\frac{x^2}{2}}dx\,.$$

(7)

Demonstration

Let k be a possible value of S_n such that $S_n=k$ it means

$$\frac{S_n-np}{\sqrt{npq}}=x_{n,k}$$

according to the relation (1). Then the probability of the event from the right of the formula (7) is

$$\sum_{a<x_{n,k}<b}P(\{S_n=k\})=\sum_{a<x_{n,k}<b}C_n^kp^kq^{n-k}$$

Given the fact that

$$x_{n,k+1}-x_{n,k}=\frac{k+1-np}{\sqrt{npq}}-\frac{k-np}{\sqrt{npq}}==\frac{1}{\sqrt{npq}}$$

we obtain

$$\sum_{a<x_{n,k}<b}P(\{S_n=k\})\sim$$

$$\sim\frac{1}{\mathcal{K}}\sum_{a<x_{n,k}<b}e^{-\frac{x^2}{2}}(x_{n,k+1}-x_{n,k})\qquad(8)$$

The correspondence between k and $x_{n,k}$ is bijective and when k varies from 0 to n , $x_{n,k}$ varies in interval $\left[-\sqrt{\frac{np}{q}},\sqrt{\frac{np}{q}}\right]$, not continuous, with step

$$x_{n,k+1}-x_{n,k}=\frac{1}{\sqrt{npq}}.$$

For n sufficient large, the interval will contain $[a,b)$, and the points $x_{n,k}$ will be in entire of $[a,b)$, dividing it into equidistant intervals of length $\frac{1}{\sqrt{npq}}$.

We assume that the lowest and highest value of k that satisfy the conditions $a\le x_{n,k}<b$ are, respectively, j and l and we will have

$x_{j-1} < a < x_j < x_{j+1} < \cdots < x_{l-1} < x_l < b < x_{l+1},$
and the sum from (2) can be written

$$\sum_{k=j}^{l} \varphi(x_{n,k})(x_{n,k+1} - x_{n,k})$$

where

$$\varphi(x) = \frac{1}{\mathcal{K}} e^{-\frac{x^2}{2}}.$$

This is the Riemann's sum for definited integral

$$\int_a^b \varphi(x)dx.$$

Making $n \to \infty$, the division becomes more and more fine and the sum converges to the given integral. It remains to determine the constant $\mathcal{K}$ In the formula

$$\lim_{n\to\infty} P\left(\left\{a \le \frac{S_n - np}{\sqrt{npq}} \le b\right\}\right) == \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx$$

(9)

$$\lim_{n\to\infty} P\left(\left\{-b \le \frac{S_n - np}{\sqrt{npq}} \le b\right\}\right) == \frac{1}{\mathcal{K}} \int_{-b}^{b} e^{-\frac{x^2}{2}} dx$$

(10)

If we note

$$X = \frac{S_n - np}{\sqrt{npq}},$$

then

$$M[X] = 0,$$

$$D^2(X) = 1$$

and we obtain

$$P\left(\left|\frac{S_n - np}{\sqrt{npq}} \le b\right|\right) \ge 1 - \frac{1}{b^2}. \qquad (11)$$

Combining the relations (9) and (10) we will obtain the relation

$$1 - \frac{1}{b^2} \le \frac{1}{\mathcal{K}} \int_{-b}^{b} e^{-\frac{x^2}{2}} \le 1$$

and making $b \to \infty$ we get

$$\mathcal{K} = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}} dx = \sqrt{2\pi},$$

so the constant $\mathcal{K}$ from the Stirling's formula is $\sqrt{2\pi}$.

Before giving a new wording to DeMoivre-Laplace theorem, we will introduce the notion of *convergence in distribution*.

Definition 1. *Let $\{X_n\}_{n\in\mathbb{N}}$ be a random variables string and $\{F_n\}_{n\in\mathbb{N}}$ be the string corresponding distribution functions. If the string distribution functions $\{F_n\}_{n\in\mathbb{N}}$ converges to a distribution function F in all the continuity point x_0 of the distribution function F corresponding to the random variable X, ie if*

$$\lim_{n\to\infty} F_n(x_0) = F(x_0)$$

*then we will say that the string $\{X_n\}_{n\in\mathbb{N}}$ **converges in distribution** to X and it is noted*

$$X_n \xrightarrow{rep} X.$$

Remark. Because there may be more random variables with the same distribution function, it results from definition that the limit of a range of random variables converges in distribution is not unique.

Remark. An example of such convergence is the following: if $\{X_n\}_{n\in\mathbb{N}}$ is a string of random variables distributed

$Bi\left(n, \frac{\lambda}{n}\right)$ then $\{X_n\}_{n\in\mathbb{N}}$ converges in distribution to the random variable X distributed Poisson with parameter λ. (Relation between binomial distribution and Poisson distribution).

We now give a more general DeMoivre-Laplace theorem. We note with

$$S_n = X_1 + X_2 + \cdots + X_n, n \ge 1$$

where $X_j, j = \overline{1, n}$ are independent Bernoulli random variables. We know that

$$M[X_j] = p,$$
$$D^2[S_n] = npq, \quad j = \overline{1, n},$$

and for every n,

$$M[S_n] = np, \quad D^2[S_n] = npq.$$

Note

$$X_j^* = \frac{X_j - M[X_j]}{D[X_j]},$$

$$S_n^* = \frac{S_n - M[S_n]}{D[S_n]} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} X_j^* \qquad (11)$$

S_n^* is a random variable and is called a normalized random variable. We have for every j and n

$$M[X_j^*] = 0 \, , D^2[X_j^*] = 1$$

The liniar transformation that leads X_j in X_j^* or S_n in S_n^* aims to bring them to a random variable with mean 0 and variance 1. Every S_n^* is a random variable which takes as values

$$x_{n,k} = \frac{k - np}{\sqrt{npq}}$$

This is just $x_{n,k}$ from DeMoivre-Laplace theorem and

$$P(\{S_n^* = x_{n,k}\}) = C_n^k p^k q^{n-k}, 0 \le k \le n$$

If we use the corresponding distribution function

$$P(\{S_n^* < x\}) = F_n(x)$$

and if F is the standard normal distribution function, then DeMoivre–Laplace theorem can be written in a form that is more elegant

$$\lim_{n\to\infty} F_n(x) = F(x)$$

Remark. The theorem can be extended in the following sense: let $\{X_n\}_{n\in\mathbb{N}}$ a string of independent random variables with the same distribution that does not need to be specified. It should, instead that $M[X_j] = m < \infty$ and $D^2[X_j] = \sigma^2 < \infty$. The Laplace theorem occurs also in these conditions.

Application. We will now determine how likely wrong betting options, 4 players of 15, each playing independently during a game, where the probability that a player has bet wrong is $p = 0{,}3$.

The bett is a random variable with binomial distribution:

$$X: \begin{pmatrix} k \\ C_n^k (0{,}3)^k (0{,}7)^{15-k} \end{pmatrix}, k = \overline{0{,}15}$$

$$P_{15;4} = C_{15}^4 (0{,}3)^4 (0{,}7)^{11} = \frac{15!}{4!\,11!} \cdot 0{,}0081 \cdot 0{,}05764801 = 0{,}2186$$

With the help of DeMoivre-Laplace theorem we can approximate

$$x_{15,4} = \frac{4 - \frac{15}{4}}{\sqrt{15 \cdot 0{,}3 \cdot 0{,}7}} = 0{,}1408$$

$$C_{15}^4 (0{,}3)^4 (0{,}7)^{11} \sim \frac{1}{\sqrt{2\pi \cdot 0{,}3 \cdot 0{,}7 \cdot 15}} e^{-\frac{(0{,}1408)^2}{2}} \approx 0{,}2225$$

Theorem [(3)] For the sums S_n in generalized conditions above and $a < b$ there we have

$$\lim_{n\to\infty} P\left(\left\{a < \frac{S_n - nm}{\sigma\sqrt{n}} \le b\right\}\right) == \frac{1}{\sqrt{2\pi}} \int_a^b e^{-\frac{x^2}{2}} dx \, . \qquad (12)$$

The general formulation of laws limit arises in the following way:

Let $(X_n)_{n\in\mathbb{N}^*}$ a string of random variables. If there exists two strings of real number $(a_n)_{n\in\mathbb{N}^*}$ and $(b_n)_{n\in\mathbb{N}^*}$ such that

$\frac{X_n - a_n}{b_n} \xrightarrow{B} X$, where X has a determined distribution law, then distributions thus obtained constitutes a family that we call family of type L distributions, in which the normal law occupies a very important place.

Theorem [(4)]. Central limit theorem (Lindeberg-Levy)

Let $(X_n)_{n\in\mathbb{N}^*}$ be a string of independent random variables, identically distributed, admitting moments of order one and two. If we consider the string of random variables $(Y_n)_{n\in\mathbb{N}^*}$

$$Y_n = \frac{\sum_{k=1}^n X_k - M(\sum_{k=1}^n X_k)}{D(\sum_{k=1}^n X_k)}$$

then

$$Y_n \xrightarrow{B} X \in N(0;1),$$

$$\lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} P(\omega : Y_n(\omega) < x) = \frac{1}{2\pi} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy.$$

Demonstration

It is noted immediately that

$$M\left(\sum_{k=1}^n X_k\right) = \sum_{k=1}^n M(X_k) = nm \, ;$$

$m = M(X_k)\,, k = 1,2,\dots$

$$D^2\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} D^2(X_k) = n\sigma^2\,;$$

$\sigma^2 = D^2(X_k)\,, k = 1,2,\dots$

and, so,

$$Y_n = \frac{\sum_{k=1}^{n}(X_k - m)}{\sqrt{n}\sigma}$$

Then.

$$\varphi_{Y_n}(t) = M\left(e^{itY_n}\right) = M\left(e^{it\frac{\sum_{k=1}^{n}(X_k-m)}{\sqrt{n}\sigma}}\right) =$$

$$= M\left(\prod_{k=1}^{n} e^{i\frac{t}{\sqrt{n}\sigma}(X_k-m)}\right) =$$

$$= \prod_{k=1}^{n} M\left(e^{i\frac{t}{\sqrt{n}\sigma}(X_k-m)}\right) =$$

$$= \prod_{k=1}^{n} \varphi_{(X_k-m)}\left(\frac{t}{\sqrt{n}\sigma}\right) = \left(\varphi\left(\frac{t}{\sqrt{n}\sigma}\right)\right)^n$$

Because for every $t \in \mathbb{R}$, if n is sufficient large, $\left|\frac{t}{\sqrt{n}\sigma}\right| < 1$, then we can extend in series around the origin, the function φ and we will obtain

$$\varphi\left(\frac{t}{\sqrt{n}\sigma}\right) = 1 - \frac{\sigma^2}{2!}\frac{t^2}{\sigma^2 n} + \theta\left(\frac{1}{n^{\frac{3}{2}}}\right) =$$

$$= 1 - \frac{t^2}{2n} + \theta\left(\frac{1}{n^{3/2}}\right)$$

It follows that

$$\varphi_{Y_n}(t) = \left(1 - \frac{t^2}{2n}(1+\varepsilon_n)\right)^n,$$

$$\lim_{n\to\infty} \varepsilon_n = 0$$

and

$$\lim_{n\to\infty} \varphi_{Y_n}(t) = e^{-\frac{t^2}{2}}$$

From the uniqueness and inversion theorem it results that

$$\lim_{n\to\infty} F_{Y_n}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy.$$

Theorem [(5)] (Lyapunov) *Let $(X_n)_{n\in\mathbb{N}^*}$ be a string of independent random variables for which there exist*

$M(X_k) = m_k\,, D^2(X_k) = D_k^2\,, M(|X_k - m_k|^3) = H_k^3, k \in \mathbb{N}^*.$

We note

$$S_n = \left(\sum_{k=1}^{n} D_k^2\right) \qquad H_n = \left(\sum_{k=1}^{n} H_k^3\right)^{1/3}$$

If

$$\lim_{n\to\infty} \frac{K_n}{S_n} = 0$$

then

$$Y_n = \frac{\sum_{k=1}^{n} X_k - M\left(\sum_{k=1}^{n} X_k\right)}{D\left(\sum_{k=1}^{n} X_k\right)} \xrightarrow{B} X \in N(0,1)$$

ie

$$\lim_{n\to\infty} F_{Y_n}(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}\,dy.$$

Demonstration

We will use also the characteristic function method; as

$$D^2\left(\sum_{k=1}^{n} X_k\right) = \sum_{k=1}^{n} D^2(X_k) = S_n^2$$

The random variable Y_n can be also written in the following form

$$Y_n = \frac{\sum_{k=1}^{n}(X_k - m_k)}{S_n}$$

and, with this,

$$\varphi_{Y_n}(t) = M\left(e^{itY_n}\right) = M\left(e^{i\frac{t}{S_n}\sum_{k=1}^{n}(X_k-m_k)}\right) =$$

$$= M\left(\prod_{k=1}^{n} e^{i\frac{t}{S_n}\sum_{k=1}^{n}(X_k-m_k)}\right) =$$

$$= \prod_{k=1}^{n} M\left(e^{i\frac{t}{S_n}\sum_{k=1}^{n}(X_k-m_k)}\right)$$

So,

$$\varphi_{Y_n}(t) = \prod_{k=1}^{n} \varphi_{X_k - m_k}\left(\frac{t}{S_n}\right)$$

But,

$$\varphi_{X_k - m_k}(t) = e^{-itm_k}\varphi_{X_k}(t) = a_k(t) + ib_k(t)$$

If we note with $G_k(x)$ the distribution function of the random variable $X_k = m_k$, it results that:

$$a_k(t) = \int_{-\infty}^{\infty} \cos txdG_k(x) =$$

$$= 1 - \frac{D_k^2 t^2}{2} + \frac{t^3}{6}\int_{-\infty}^{\infty} \theta_1 x^3\, dG_k(x)$$

$$b_k(t) = \int_{-\infty}^{\infty} \sin txdG_k(x) =$$

$$= 1 - \frac{D_k^2 t^2}{2} + \frac{t^3}{6}\int_{-\infty}^{\infty} \theta_2 x^3\, dG_k(x)$$

with $|\theta_j| < 1, j = 1,2.$

Then,

$\varphi_{X_k - m_k}(t) = 1 - \frac{D_k^2 t^2}{2} + t^3 R_k$, where

$$|R_k| = \frac{1}{6}\left|\int_{-\infty}^{\infty}(\theta_1 x^3 - \theta_2 x^3)dG_k(x)\right| \leq$$

$$\leq \frac{1}{6}\int_{-\infty}^{\infty}|x^3||\theta_1 - \theta_2|dG_k(x) \leq \frac{H_k^3}{3}$$

and, from here,

$$\varphi_{X_k - m_k}\left(\frac{t}{S_n}\right) = 1 - \frac{D_k^2 t^2}{2S_n^2} + \frac{t^3}{3S_n^3}$$

and

$$\ln \varphi_{Y_n}(t) = \sum_{k=1}^{n} \ln \varphi_{X_k - m_k}\left(\frac{t}{S_n}\right) =$$

$$= \sum_{k=1}^{n} ln\left(1 - \frac{D_k^2 t^2}{2S_n^2} + \frac{t^3}{3S_n^3}\right).$$

Because $\frac{K_n}{S_n} \to 0$ when $n \to \infty$, it results that for every

$\varepsilon > 0$, there exist a rank $N(\varepsilon)$ such that for every

$n > N(\varepsilon)$ we will have $\frac{K_n}{S_n} < \frac{\varepsilon}{|t|}$, $t \neq 0$.

From here it results that

$$\frac{H_k^3}{S_n^3} < \frac{\varepsilon^3}{|t|^3}$$

if $n > N(\varepsilon)$.

From the Lyapunov's inequality (the monotony of absolute moments) we have $D_k \leq H_k, k \in \mathbb{N}^*$ and, so,

$$\frac{D_k^2}{S_n^2} \leq \frac{H_k^2}{S_n^2} = \left(\frac{H_k^3}{S_n^3}\right)^{2/3} \leq \frac{H_k^2}{S_n^2} \leq \frac{\varepsilon^2}{t^2}, k = 1,2,\dots,n$$

Then, for every $\varepsilon > 0$,

$$\left|-\frac{D_k^2 t^2}{2S_n^2} + \frac{t^3 R_k}{3S_n^3}\right| < \frac{\varepsilon^2}{2} + \frac{\varepsilon^3}{3} < \varepsilon^2$$

We put

$$\ln \varphi_{Y_n}(t) = \sum_{k=1}^{n} ln\left(1 - \frac{D_k^2 t^2}{2S_n^2} + \frac{t^3 R_k}{3S_n^3}\right)$$

In the form

$$\ln \varphi_{Y_n}(t) + \frac{t^2}{2} =$$

$$= \sum_{k=1}^{n}\left[\ln\left(1 - \frac{D_k^2 t^2}{2S_k^2} + \frac{t^3 R_k}{3S_n^3}\right) + \frac{D_k^2 t^2}{2S_n^2}\right].$$

Such that $|\ln(1+x) - x| \leq |x^2|$ if $|x| \leq \frac{1}{2}$, with notation

$A_k = -\frac{D_k^2 t^2}{2S_n^2}, B_k = \frac{t^3 R_k}{3S_n^3}$, we can write

$$\left|\ln \varphi_{Y_n}(t) + \frac{t^2}{2}\right| =$$

$$= \left|\sum_{k=1}^{n} \ln(1 + A_k + B_k) - (A_k + B_k) + B_k\right| \leq$$

$$\leq \sum_{k=1}^{n}|A_k + B_k|^2 + \sum_{k=1}^{n}|B_k|.$$

But

$$\sum_{k=1}^{n} |B_k| \le \frac{|t|^3 K_n^3}{3S_n^3} \le \frac{\varepsilon^3}{3}$$

also

$$\sum_{k=1}^{n} |A_k + B_k|^2 \le \varepsilon^2 \sum_{k=1}^{n} (|A_k| + |B_k|) \le$$

$$\le \varepsilon^2 \frac{|t|^2}{2} + \frac{\varepsilon^5}{3}$$

From here, it follows that

$$\left|\ln \varphi_{Y_n}(t) + \frac{t^2}{2}\right| \le \varepsilon^2 \frac{|t|^2}{2} + \frac{\varepsilon^5}{3} + \frac{\varepsilon^3}{3} < \varepsilon$$

if $\varepsilon < \frac{1}{3|t|^2}$

So,

$$\lim_{n\to\infty} \varphi_{Y_n}(t) = e^{-\frac{t^2}{2}}$$

and from theorem of convergence of characteristic functions, it follows that

$$\lim_{n\to\infty} F_{Y_n}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy.$$

Remark. If the variables of the string $(X_n)_{n\in\mathbb{N}^*}$ are identically distributed, then

$$D_k^2 = \sigma^2;\ H_k^3 = H^3, k \in \mathbb{N}^*$$

and

$$S_n = \left(\sum_{k=1}^{n} D_k^2\right)^{1/2} = \sigma\sqrt{n}$$

$$K_n = \left(\sum_{k=1}^{n} H_k^3\right)^{1/3} = H\sqrt[3]{n}$$

It follows that

$$\frac{K_n}{S_n} = \frac{H}{\sigma} n^{-1/6} \xrightarrow[n\to\infty]{} 0$$

ie it satisfies the requirement of Lyapunov.

Definition. *We say that the string of independent random variables* $(X_n)_{n\in\mathbb{N}^*}$ *verify the* ***„L" condition (Lindeberg condition)*** *if, for every* $\varepsilon > 0$*, it follows the relation*

$$(L) \quad \lim_{n\to\infty} \alpha_n(\varepsilon) =$$

$$= \lim_{n\to\infty} \frac{1}{S_n^2} \sum_{k=1}^{n} \int_{\{x;|x-m_k|>\varepsilon S_n\}} (x-m_k)^2 dF_k(x) = 0$$

where

$$F_k(x) = P(\{\omega; X_k(\omega) < x\}).$$

Theorem [(7)] (Lindeberg-Feller) *Let* $(X_n)_{n\in\mathbb{N}^*}$ *a string of independent random*

$$\lim_{n\to\infty} F_{Y_n}(x) = \Phi(x), x \in \mathbb{R}$$

and

$$\lim_{n\to\infty} \max_{1\le k\le n} \frac{\sigma_k^2}{S_n^2} = 0$$

if and only if

$$\lim_{n\to\infty} \frac{1}{S_n^2} \sum_{k=1}^{n} \int_{\{x;|x-m_k|>\varepsilon S_n\}} (x-m_k)^2 dF_k(x) = 0$$

(is satisfied the „L" condition).

We highlight some direct consequences of Lindeberg Feller theorem.

Consequence [(7)] *If the random variables that compose the string of independent variables* $(X_n)_{n\in\mathbb{N}^*}$ *are identically distributed, then*

$$\lim_{n\to\infty} F_{Y_n}(x) = \Phi(x), \forall x \in \mathbb{R}$$

Demonstration

In this case,

$M(X_k) = m$, $D^2(X_k) = \sigma^2, k \in \mathbb{N}^*$ and, so,

$$S_n = \sigma\sqrt{n}.$$

With these, the Lindeberg's condition become

$$\alpha_n(\varepsilon) = \sum_{k=1}^{n} \frac{1}{n\sigma^2} \int_{\{x;|x-m|>\varepsilon\sigma\sqrt{n}\}} (x-m)^2 dF(x) =$$

$$= \frac{1}{n\sigma^2} n \int_{\{x;|x-m|>\varepsilon\sigma\sqrt{n}\}} (x-m)^2 dF(x)$$

and, so, $\lim_{n\to\infty} \alpha_n(\varepsilon) = 0$

ie Lindeberg's condition is accomplished.

Consequence [(8)] *If the string of the independent random variables* $(X_n)_{n\in\mathbb{N}^*}$ *has the property that the random variables* X_n *are uniformly bounded, and admits finite dispersions*

$\lim_{n\to\infty} S_n = +\infty,$

then

$\lim_{n\to\infty} F_{Y_n}(x) = \Phi(x)$

Demonstration

Given that the random variables $X_k, k \in \mathbb{N}^*$ are uniformly bounded, it results that $(\exists) A > 0$ such that $X_k - m_k \leq A, k \in \mathbb{N}^*$. From that, it follows that:

$$\int_{\{x;|x-m_k|>\varepsilon S_n\}} (x-m_k)^2 dF(x) =$$

$$= \int_{\{x;|x_k(\omega)-m_k|>\varepsilon S_n\}} (X_k(\omega)-m_k)^2 dP(\omega) \leq$$

$$\leq A^2 P(\{\omega; |X_k(\omega)-m_k| \geq \varepsilon S_n\})$$

Since

$\lim_{n\to\infty} S_n = +\infty,$

we can take n sufficient large such that $\varepsilon S_n > A$, and in this case,

$$P(\{\omega; |X_k(\omega)-m_k| \geq \varepsilon S_n\}) = 0$$

and

$$\int_{\{|x-m_k|>\varepsilon S_n\}} (x-m_k)^2 dF(x) = 0, \quad k \in \mathbb{N}^*$$

which implies checking the condition „L”.

Given the Lindeberg-Feller central limit theorem, we can easily prove the theorem of Lyapunov.

Theorem [(9)] (Lyapunov) *Let the string of the independent random variables* $(X_n)_{n\in\mathbb{N}^*}$. *If there exists* $\eta > 0$ *such that:*

$$\lim_{n\to\infty} \beta_n(\eta) = \lim_{n\to\infty} \frac{1}{S_n^{2+\eta}} \sum_{k=1}^{n} M(|x_k - m_k|^{2+\eta}) = 0$$

then

$\lim_{n\to\infty} F_{Y_n}(x) = \Phi(x), \forall x \in \mathbb{R}$

Demonstration

We verify if the „L” condition can be checked:

$$\alpha_n(\varepsilon) =$$

$$= \frac{1}{S_n^2} \sum_{k=1}^{n} \int_{\{|x-m_k|>\varepsilon S_n\}} (x-m_k)^2 \frac{\varepsilon^\eta s_n^\eta}{\varepsilon^\eta s_n^\eta} dF_k(x)$$

$$= \frac{1}{\varepsilon^\eta s_n^{2+\eta}} \sum_{k=1}^{n} \int_{\{|x-m_k|>\varepsilon S_n\}} (x-m_k)^{2+\eta} dF_k(x) \leq \frac{1}{\varepsilon^\eta} \beta_n(\eta)$$

Passing to the limit,

$$0 \leq \lim_{n\to\infty} \alpha_{n(\varepsilon)} \leq \frac{1}{\varepsilon^\eta} \lim_{n\to\infty} \beta_n(\eta) = 0$$

ie the condition „L”is satisfied.

For $\eta = 1$ is obtained exactly the formulation of Lyapunov's theorem directly demonstrated previously.

If the the string of independent random variables $(X_n)_{n\in\mathbb{N}^*}$ are

$H_k^3 = M(|x_k - m_k|^3), k \in \mathbb{N}^*$

and if

$$\lim_{n\to\infty} \frac{K_n}{S_n} = 0$$

where

$$K_n = \left(\sum_{k=1}^{n} H_k^3\right)^{1/3}$$

then

$$\lim_{n\to\infty} F_{Y_n}(x) = \Phi(x), \forall x \in \mathbb{R}.$$

3. THE MULTICRITERIAL ANALYSIS

In this part of the present article, the authors have proposed to analyze the 9 theorems presented in previous side from the point of three ways.

We have the following variants:

- variant (a): the extent of which one applies more frequently theorems
- variant (b): the difficulty to apply the 9th theorems
- variant (c): the reliable of application of the 9th theorems

9 criteria have been chosen:

1.	**Local theorem Moivre-Laplace [(1)]**
2.	**Moivre-Laplace theorem [(2)]**
3.	**Theorem [(3)]**
4.	**Central limit theorem Lindeberg-Levy [(4)]**
5.	**Lyapunov theorem [(5)]**
6.	**Lindeberg-Feller theorem [(6)]**
7.	**Consequence [(7)]**
8.	**Consequence [(8)]**
9.	**Lyapunov theorem [(9)]**

Fig. 1

Based bet on score, weighting of the criteria resulted as follows:

	[(1)]	[(2)]	[(3)]	[(4)]	[(5)]	[(6)]	[(7)]	[(8)]	[(9)]	Points	Level	γ_i
[(1)]	**1/2**	1	0	1/2	0	0	0	0	0	2	8	0,2
[(2)]	1	**1/2**	0	1/2	1/2	1/2	0	1/2	0	3,5	7,5	0,7
[(3)]	1	1	**1/2**	1/2	0	1/2	0	0	0	3,5	7,5	0,7
[(4)]	1	1	1/2	**1/2**	1/2	1	1/2	0	0	5	5	1,7
[(5)]	1/2	0	1/2	1	**1/2**	1/2	1/2	1/2	1/2	4,5	6	1,4
[(6)]	1/2	1/2	0	1	1	**1/2**	1/2	1/2	1	5,5	4	2,2
[(7)]	1	1	1/2	1/2	1/2	1	**1/2**	1	1/2	6,5	3	3,1
[(8)]	1	1	1/2	1/2	1	1	1/2	**1/2**	1	7	2	3,9
[(9)]	1	1/2	1/2	1	1	1	1	1	**1/2**	7,5	1	4,7

Fig. 2

It is noted that the main diagonal of the array contains only quadratic criteria for scoring 1/2 values because no criteria may be more important or less important than the criteria itself.

The γ_i weighting coefficients can be calculated with different formulas. We chose to use FRISCO practice formula (empirical formula given by a renowned creative group from San Francisco – US) that has been recognized worldwide as being the most performance and is long used.

Therefore, with,

$$\gamma_i = \frac{p + \Delta p + m + 0,5}{-\Delta p' + \frac{N_{crt}}{2}}$$

where

- p is the sum of points obtained (on line) of the considered element
- Δp the difference between the score of the considered element and the score at the top level element; if the element taken into account is the one located on the top floor, results Δp with the value 0
- m the number of outclassed criteria (exceeded from terms of score) the by the criteria taken into account
- N_{crt} the number of considered criterion
- $\Delta p'$ the difference between the score of the considered element and the score of the first element (resulting with a negative value); taken into account if the item is located on the first level, $\Delta p'$ results with the value 0

We obtain

$$\gamma_{(1)} = \frac{2 + (2-2) + 0 + 0,5}{-(2-7,5) + \frac{9}{2}} = \frac{2,5}{10} = 0,2$$

$$\gamma_{(2)} = \frac{3,5 + (3,5-2) + 1 + 0,5}{-(3,5-7,5) + \frac{9}{2}} = \frac{6,5}{8,5} = 0,7 = \gamma_{(3)}$$

$$\gamma_{(4)} = \frac{5 + (5-2) + 4 + 0,5}{-(5-7,5) + \frac{9}{2}} = \frac{12,5}{7} = 1,7$$

$$\gamma_{(5)} = \frac{4,5 + (4,5-2) + 3 + 0,5}{-(4,5-7,5) + \frac{9}{2}} = \frac{10,5}{7,5} = 1,4$$

$$\gamma_{(6)} = \frac{5,5 + (5,5-2) + 5 + 0,5}{-(5,5-7,5) + \frac{9}{2}} = \frac{14,5}{6,5} = 2,2$$

$$\gamma_{(7)} = \frac{6,5 + (6,5-2) + 6 + 0,5}{-(6,5-7,5) + \frac{9}{2}} = \frac{17,5}{5,5} = 3,1$$

$$\gamma_{(8)} = \frac{7 + (7-2) + 7 + 0{,}5}{-(7-7{,}5) + \frac{9}{2}} = \frac{19{,}5}{5} = 3{,}9$$

$$\gamma_{(9)} = \frac{7{,}5 + (7{,}5-2) + 8 + 0{,}5}{-(7{,}5-7{,}5) + \frac{9}{2}} = \frac{21{,}5}{4{,}5} = 4{,}7$$

According to the criteria were the following notes for each variant N_i.

	Variant (a)	Variant (b)	Variant (c)
Criteria	N_i	N_i	N_i
(1)	10	3	9
(2)	5	2	6
(3)	2	9	4
(4)	9	6	5
(5)	7	4	8
(6)	6	10	7
(7)	4	7	3
(8)	3	8	2
(9)	8	5	10

Fig. 3

It may take into account different weight now and consequence of each criterion, complementing and enhancing the table above notes (lines) with the coefficient of importance:

4. CONCLUSIONS

Multicriterial analysis technique is useful in the composition of an ranking, while qualitatively and quantitatively, of product variants, objects, methods, models, equipment, structures, creations, etc. A first valence would be that the result of such analysis in order not only put options, but it quantifies in value terms.

Rankings, to a large extent, have a high degree of subjectivity and aims the most of the time only the qualitative aspect. Multicriterial analysis technique gives, from the viewpoint of its user, results found to a great extent objectives (ie, this technique objectifies in an important measure the results).

It is noted that after the ranking did, the approximation theorems studied in this paper are preferred to be taken in the variant (b).

We want to emphasize that point III is our own creation. Efforts have been made to develop this multicriterial analysis applied to approximation theorems studied in this paper, hoping that we will develop it in the future.

		Variant (a)		Variant (b)		Variant (c)	
Criteria	γ_i	N_i	$N_i \times \gamma_i$	N_i	$N_i \times \gamma_i$	N_i	$N_i \times \gamma_i$
(1)	0,2	10	2	3	0,6	9	1,8
(2)	0,7	5	3,5	2	1,4	6	4,2
(3)	0,7	2	1,4	9	6,3	4	2,8
(4)	1,7	9	15,3	6	10,2	5	8,5
(5)	1,4	7	9,8	4	5,5	8	11,2
(6)	2,2	6	13,2	10	22	7	15,4
(7)	3,1	4	12,4	7	21,7	3	9,3
(8)	3,9	3	11,7	8	31,2	2	7,8
(9)	4,7	8	37,6	5	23,5	10	47
Final ranking			106,9		122,5		108

Fig. 4

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