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SCHWARZ METHOD FOR VARIATIONAL INEQUALITIES

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Abstract: *In this paper we want to bring into question the Schwarz overlapping domain decomposition method, taking into account that domain decomposition is a technique where the original domain is decomposed into a set of smaller sub- domains. We will talk about the additive Schwarz method for variational inequalities, presenting first the general framework where we expose the problem that we want to study. The purpose of this work s is to exploit a convergence theory for the specified method. The convergence results from the norm estimates for some error reduction operators. The additive Schwarz algorithm is formulated in a way which admits a nice recurrence for the errors between two consecutive steps. Through a study for projection operators onto closed and convex subsets of a Hilbert space, we will demonstrate a geometric convergence for our method. We have to mention that for simplicity, the theory will be demonstrated only for the obstacle problem.*

Keywords: *domain decomposition, admissible decomposition, error estimates*

1. Introduction

The additive Schwarz method, named after H. A. Schwarz, solves a boundary value problem for a partial differential equation approximately by splitting it into boundary value problems on smaller domains and adding the results.

The paper is organized as follows: firstly we will study the variational inequalities in an abstract framework, then the general result developed before will be applied to an obstacle problem in Sobolev spaces.

2. The Schwarz method for variational inequalities

2.1. General framework An iterative scheme

Let $(V,(\cdot,\cdot))$ be a Hilbert space and $a: V\times V\to \mathbf{R}$ a bilinear, symmetric, coercive and continuous form and $K\subset V$ a convex, closed subset. We consider the following variational inequality:

$$\begin{cases} u\in K \\ a(u,v-u)\geq f(v-u), \forall\, v\in K \end{cases},$$

(PI)

where f is a linear continuous functional on V (i.e. $f\in V'$).

From the properties of the bilinear form $a(\cdot,\cdot)$ it results that $a(u,v)\cong(u,v), \forall u,v\in V$.

Furthermore, we have $a(u,v)=(u,v), \forall u,v\in V$. Let be $F: V\to \mathbf{R}$.

It is known that the problem (P1) is equivalent to the following minimization problem: $u \in K, F(u) \leq F(v), \forall v \in K$.

(P2)

We want to approximate the solution of (P1) by iterative procedures. Then, let V_i, $i = \overline{1,m}$ be subspaces of V such that $V = \sum_{i=1}^{m} V_i$. The interest is to define an algorithm for constructing a sequence $(u_n)_{n \in \mathbb{N}}$ to approximate the exact solution of the problem (P1), which is the minimum of the functional F . It is natural to impose that the solution from the step $n+1$ to decrease the value of the functional F, i.e. $F(u_{n+1}) \leq F(u_n)$.

Algorithm description

We proceed in two steps.

1. It is defined $u_{n,i} \in V_i$ such that:

$$F(u_n + u_{n,i}) \leq F(u_n + v_i), \forall v_i \in K_{n,i}, \qquad \text{(P3)}$$

where $K_{n,i} = \{v_i \in V_i | \ u_n + v_i \in K\}$.

2. It is defined

$$u_{n+1} = u_n + \rho \sum_{i=1}^{m} u_{n,i}, \qquad (*)$$

with ρ chosen such that $u_{n+1} \in K$.

Let be $\mu = \rho m \leq 1$. We have:

$$u_{n+1} = u_n + \rho \sum_{i=1}^{m} u_{n,i} = (1-\mu)u_n + \mu \sum_{i=1}^{m} \frac{1}{m}(u_n + u_{n,i}).$$

Since $u_n \in K$ and $\sum_{i=1}^{m} \frac{1}{m}(u_n + u_{n,i}) \in K$, we observe that a sufficient condition to have $u_{n+1} \in K$ is that $\mu \leq 1$, i.e. $\rho \leq \frac{1}{m}$.

Obviously, the formulation of the problem (P3) is equivalent to the following variational inequality:

$$\begin{cases} u_{n,i} \in K_{n,i} \\ a(u_n + u_{n,i}, v_i - u_{n,i}) \geq f(v_i - u_{n,i}), \forall v_i \in K_{n,i} \end{cases} \qquad \text{(P4)}$$

Furthermore, we will make the following assumption which is necessary to demonstrate the convergence:

Assumption 2.1.

The problem (P4) is equivalent to the following problem:

$$\begin{cases} u_{n,i} \in K_{n,i} \\ a(u_n + u_{n,i}, v_i - u_{n,i}) \geq a(u, v_i - u_{n,i}), \forall v_i \in K_{n,i} \end{cases} \qquad \textit{(P5)}$$

We can write the problem (P5) under the form:

$$\begin{cases} u_{n,i} \in K_{n,i} \\ a(u_{n,i}, v_i - u_{n,i}) \geq a(u - u_n, v_i - u_{n,i}), \forall v_i \in K_{n,i} \end{cases} \qquad \textit{(P6)}$$

The correction is given by the solving the problem (P6).

Let $P_{n,i}: V_i \to K_{n,i}$ be the projection operator on the convex closed set $K_{n,i}$. From (P6) it results that:

$$u_{n,i} = P_{n,i}(u - u_n) \qquad (**)$$

With these preliminary the iterative scheme is defined as follows:

Algorithm 2.1.

Let be $u_0 \in K$. We compute the sequence of approximations $\{u_n\}$ as follows:

1. *We compute $u_{n,i}$ from the problem (P4).*
2. *We compute u_{n+1} from (*).*
3. Let $e_n = u - u_n$ be the error at the step n . From (**) it results that:

$u_{n,i} = P_{n,i} e_n$.

Thus, from (*) it results that:

$e_{n+1} = (I - \rho T_n) e_n$,

where T_n is the additive operator

$$T_n = \sum_{i=1}^{m} P_{n,i}.$$

To demonstrate the convergence of the Schwarz method, we analyse the additive operator T_n.

2.2. Technical estimates

Let $K_i \subset V, i = \overline{1,m}$ be convex closed subsets such that $0 \in K_i, \forall i = \overline{1,m}$. We observe that this hypothesis is satisfied for $K_i = K_{n,i}$ because $u_n \in K$. Let be $f \in V$. We consider the problem (P1) in the case $a(\cdot,\cdot) = (\cdot,\cdot)$, which is equivalent to $u = P_i f$, where $P_i : V \to K_i$ is the projection operator, or we can have:

$$\begin{cases} P_i f \in K_i \\ ([P]_i f, v - P_i f) \geq (f, v - P_i f), \end{cases}$$
$$\forall v \in K_i. \qquad (2.1)$$

The corresponding additive operator is given by:

$$T = \sum_{i=1}^{m} P_i \, . \qquad (A)$$

Taking $v = 0$ in (2.1) we obtain:

$$\|P_i f\|^2 = (P_i f, P_i f) \leq (f, P_i f),$$
$$\forall f \in V. \qquad (2.2)$$

From (2.1) we obtain:

$$(f, v - P_i f) \leq ([P]_i f, v), \forall v \in K_i.$$
(2.3)

Definition 2.1: A vector $f \in V$ is said to have an admissible decomposition with respect to $\{K_i\}$ and a fixed constant C_0, if there exists a partition of f:

$$f = f_1 + f_2 + \cdots + f_m, f_i \in K_i,$$

such that

$$\sum_{i=1}^{m} \|f_i\|^2 \leq C_0 \|f\|^2 \, . \qquad (2.4)$$

Lemma 2.1: If $w \in V$ has an admissible decomposition with respect to $\{K_i\}$ and the constant C_0, then we have the inequality: $(f, f) \leq (2 + C_0)(f, Tf)$.

Demonstration: We have:

$$\|f\|^2 = (f,f) = (f, \sum_{(i=1)}^{m} f_i) = \sum_{(i=1)}^{m} [(f, f_i)] = \sum_{(i=1)}^{m} [(f, f_i)] - P_i f + P_i f) = \sum_{(i=1)}^{m} [(f, f_i)] - P_i f) + \sum_{(i=1)}^{m} [(f, P_i] f) \leq \sum_{(i=1)}^{m} (P_i f, f_i) + (f, T$$
,

Next, we investigate the boundedness of the operator T. Let be $C_{ij} \in [0,1]$ which satisfies the inequality:

$$|([P]_i f, P_j g)| \leq C_{ij} \|P_i f\| \|P_j g\|,$$
$$\forall f, g \in V \qquad (2.5)$$

Let be $C = (C_{ij})_{i,j=\overline{1,m}}$ and let $|C|$ be the norm of the matrix C.

Lemma 2.2: Let T be defined as above. Then $\|Tf\|^2 \leq |C|(f, Tf), \forall f \in V$.

Consequently, $\|Tf\| \leq |C| \|f\|, \forall f \in V$. (2.6)

From the above two lemmas, we easily deduce the following properties of the operator T.

Theorem 2.1: Let $f \in V$ having an admissible decomposition with respect to $[\{K]_i\}$ and the constant C_0:

$$(2 + C_0)^{-1} \|f\|^2 \leq (f, Tf) \leq$$

$\|C\| \|f\|^2.$ (B)

and

$(2 + C_0)^{-2} \|f\|^2 \leq \|Tf\|^2 \leq$
$\|C\|^2 \|f\|^2.$ (C)

Demonstration: For (B) we have:

• from lemma 2.1 we have:

$(f,f) \leq (2 + C_0)(f,Tf) \Rightarrow (2 + C_0)^{-1} \|f\|^2 \leq (f,Tf), \forall f \in V.$

•

$(f,Tf) \leq \|f\| \|Tf\| \leq \|f\| \|C\| \|f\| = \|C\| \|f\|^2, \forall f \in V$

, where we used (2.6).

For (C) we have:

• from lemma 2.1 and the Cauchy- Schwarz inequality we have

$\|f\|^2 \leq (2 + C_0)(f,Tf) \leq (2 + C_0)\|f\| \|Tf\| \Rightarrow$
$(2 + C_0)^{-2} \|f\|^2 \leq \|Tf\|^2, \forall f \in V.$

• from lemma 2.2, the relation (2.6), we have:

$\|Tf\|^2 \leq \|C\|^2 \|f\|^2, \forall f \in V$.

2.3. The convergence

Theorem 2.2: Let u_n be the solution given by algorithm 2.1 and let u be the solution of the problem (P1). We assume that the assumption 2.1 is satisfied. We also assume that $u_0 \in K$ is an element such that at each step n , $u - u_n$ has an admissible decomposition with respect to $\{K_{n,i}\}$ and a fixed constant C_0 independent of n. Then, for ρ chosen sufficiently small, $\exists \theta \in (0,1)$ such that:

$\|u - u_{n+1}\|^2 \leq \theta \|u - u_n\|^2$.

Demonstration: We know that

$e_{n-1} = (I - \rho T_n)e_n.$

It results that:

$\|e_{n+1}\|^2 = \|e_n\|^2 - 2\rho(T_n e_n, e_n) + \rho^2 \|T_n e_n\|^2.$

We use the relation (B) from theorem 2.1, i.e.:

$(2 + C_0)^{-1} \|f\|^2 \leq (f,Tf),$

stating that in our case we have

$(2 + C_0)^{-1} \|e_n\|^2 \leq (e_n, T_n e_n).$

So, $-(T_n e_n, e_n) \leq -(2 + C_0)^{-1} \|e_n\|^2$.

We also use the relation (C) from theorem 2.1. i.e.:

$\|Tf\|^2 \leq \|C\|^2 \|f\|^2,$

stating that in our case we have

$\|T_n e_n\|^2 \leq \|C\|^2 \|e_n\|^2.$

Replacing these two obtained relations in the above equality, we have:

$\|e_{n+1}\|^2 \leq [1 - 2\rho(2 + C_0)^{-1} + \rho^2 \|C\|^2] \|e_n\|^2$
, (D)

where C depends on n and $C_{ij}^n \in [0,1]$ such that:

$(P_{n,i}f, P_{n,j}g) \leq C_{ij}^n \|P_{n,i}f\| \|P_{n,j}g\|, \forall f,g \in V$
.

3.4. An application in the domain decomposition method

For simplicity, the idea will be illustrated only for obstacle problems. Let $\Omega \subset \mathbf{R}^n, n \in \mathbf{N},$ be an open bounded domain with Lipschitz continuous boundary $\Gamma = \partial\Omega.$ We assume that $\partial\Omega = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, is a partition of the boundary such that $meas(\Gamma_1) > 0.$ We consider the Sobolev space

$V = \{v \in H^1(\Omega): v = 0$ on $\Gamma_1\}$,

the convex set

$K = \{v \in V: v \geq 0$ in $\Omega\}$, (2.7)

and the problem:

$$\begin{cases} u \in K \\ a(u, v - u) \geq f(v - u), \forall v \in K, \end{cases} \quad (2.8)$$

where $a(\cdot,\cdot)$ is a symmetric, continuous and positive definite bilinear form on $V \times V$ and $f \in V', V'$ being the dual of the space V. For simplicity, the analysis can be restricted to the following bilinear form model:

$a(v,w) = \int_\Omega \nabla v \cdot \nabla w dx$ on $\Omega,$

$v,w \in V.$ (2.9)

First, we decompose the domain into overlapping sub-domains:

$$\Omega = \bigcup_{i=1}^{m} \Omega_i ,$$
(2.10)

where Ω_i are open sub-domains with Lipschitz continuous boundary.

Secondly, we define $V_i t = \{v_i t \in V : v_i t = 0$ in $\Omega - \Omega_i t\}, t = (1, m)$. Next, we apply the abstract theory that we exposed it before, to approximate the solution u of the problem (2.8).

Algorithm 2.2: Let be $u_0 \in K$. We compute the sequence of approximations $\{u_i n\}$ as follows:

1. We assume that u_n is known. We consider the convex set:

$$K_{n,i} = \{v_i \in V_i : v_i + u_n \in K\}. \qquad (2.11)$$

For each $i \in \{1, \dots, m\}$, we compute $v_{n,i}$ by solving the problem:

$$\begin{cases} v_{n,i} \in K_{n,i} \\ a(u_n + v_{n,i}, v - v_{n,i}) \geq f(v - v_{n,i}), \forall v \in K_{n,i} \end{cases} \qquad (2.12)$$

We update the approximation by:

$$u_{n+1} = u_n + \rho \sum_{i=1}^{m} v_{n,i} , \qquad (2.13)$$

where ρ should be chosen such that $u_{n+1} \in K$.

The following lemma provides a useful criterion for choosing ρ.

Lemma 2.3: For any $x \in \Omega$, let $N(x)$ be the number of sub-domains containing x. If ρ is chosen as a smooth positive function such that:

$$\rho(x) N(x) \leq 1, \forall x \in \Omega, \qquad (2.14)$$

then the approximation u_{n+1} from (2.13) is a function in the convex set K. Also, the convergence that results theorem 2.2 holds in this case if:

$$1 - 2\rho_1 (1 - C_0)^{-1} + \rho_2^{\,2} \|C\|^2 \leq \theta < 1,$$

where $\rho_1 = \min_x \rho(x)$ and $\rho_2 = \max_x \rho(x)$.

Demonstration: First, for any $x \in \Omega$, $\sum_{i=1}^{m} [v_i(n,t)(x)] \geq N(x) \min_i [v_i(n,t)](x)$.

So, from (2.13) and (2.14), we have:

$$u_{n+1}(x) \geq u_n(x) + \rho(x) N(x) \min_i v_{n,i}(x).$$

If $\min_i [v_i(n,t)](x) \geq 0$, then from the above inequality we have that $u_{n+1}(x) \geq 0$.

If $\min_i [v_i(n,t)](x) < 0$, then from (2.14) we have:

$$u_i(n+1)(x) \geq u_i n(x) + \min_i [v_i(n,t)](x) \geq 0,$$

where at the last step we have used the fact that: $u_n(x) + v_{n,i}(x) \geq 0, \forall i \in \{1, \dots, m\}$.

Regarding the second part of the lemma, we see that it results from the relation:

$$\|e_{n+1}\|^2 \leq (1 - 2\rho_1 (1 - C_0)^{-1} + \rho_2^{\,2} \|C\|^2) \|e_n\|^2$$

, instead of (D).

To demonstrate the convergence by using the abstract result established above, we first have to show that the assumption 2.1 is satisfied for the model problem (2.8).

With $u_{n,i} = u_n + v_{n,i}$, we can rewrite the problem (P5) as follows:

$$u_{n,i} \in K_{n,i} + u_n$$

$$a(u_{n,i}, v_i - u_{n,i}) \geq a(u, v_i - u_{n,i}), \forall v_i \in K_{n,i} + u_n . \qquad (2.15)$$

Then, the assumption 2.1 is equivalent to the equivalence of the problems (2.12) and (2.15).

Lemma 2.4: Let u be the solution of the problem (2.8) and $u_{n,i}$ the solution of the problem (2.15). Then, we have the statements:

1. If the approximation from the step n satisfies the conditions $u_n \in K$ and $u - u_n \in K$, then $u_{n,i} \in K$ and $u - u_{n,i} \in K$.
2. If the inequalities (2.12) and (2.15) are equivalent in the sense that:

$$\begin{cases} u_{n,i} \text{ verifies (2.15)} \\ v_{n,i} \text{ verifies (2.12),} \end{cases}$$

then $u_{n,i} = u_n + v_{n,i}$.

Demonstration: 1. Let be $\Omega_i^{+} = \{x \in \Omega_i : [u]_i(n,t) > 0\}$.

Taking $v_i = u_{n,i} \pm \varepsilon w_i$ in (2.15) we have:

$a(u_{n,i} - u, \varepsilon w_i) = 0, \forall w_i \in V_i, w_i = 0$ on $\Omega - \bar{\Omega}_i^{+}$.

Since $\varepsilon \in (0,1)$, it results that:

$$a(u_{n,i} - u, w_i) = 0, \forall w_i \in V_i, w_i = 0 \text{ on } \Omega - \bar{\Omega}_i^{+}. \quad (2.16)$$

2.We show that $u - u_{n,i} \in K$. Let be $D_i = \{x \in K : u_{n,i} - u > 0\}$.

We claim that $D_i \subset \Omega_i^{+}$.

In fact, if $x \in D_i$, then $u_{n,i}(x) - u(x) > 0$,i.e.

$$u_{n,i}(x) > u(x) \geq 0. \quad (2.17)$$

Since in $\Omega - \Omega_i$ we have $u_{n,i} = u_n \leq u$, then (2.17) involves $x \in \Omega_i$ and $u_{n,i}(x) > 0$. Thus, $x \in \Omega_i^{+}$. We observe that $u - u_n \in K$ (i.e. $u_n \leq u$) and therefore, $u_{n,i} - u = u_n - u \leq 0$ on $\partial\Omega_i \cap \Omega$.

Since $D_i \subset \Omega_i^{+}$, the function

$$\Phi_i = \begin{cases} u_{n,i}(x) - u(x), & x \in D_i \\ 0, & x \in \Omega - D_i, \end{cases}$$

is defined on V_i and vanishes in $\Omega - \bar{\Omega}_i^{+}$.

Replacing w_i in (2.16) by Φ_i we have:

$a(u_{n,i} - u, \Phi_i) = 0 \Rightarrow a(\Phi_i, \Phi_i) = 0 \Rightarrow \Phi_i = 0.$

Therefore, D_i must be the empty set. This shows that $u - u_{n,i} \geq 0$ and thus, $u - u_{n,i} \in K$.

We show now that the inequalities (2.15) and (2.12) are equivalent in the sense established in the theorem.

In fact, from (2.15) we have:

$$a(u_{n,i}, v_i - u_{n,i}) \geq a(u, v_i) - a(u, u_{n,i}),$$
$$\forall v_i \in K_{n,i} + u_n \subset K. \quad (2.18)$$

It is known that $a(u, v_i) \geq f(v_i)$, $\forall i = \overline{1, m}$ and since $u - u_{n,i} \in K$, we have:

$$-a(u, u_{n,i}) = a(u, u - u_{n,i} - u) \geq f(u - u_{n,i} - u) = f(-u_{n,i}).$$

Replacing the last two relations in (2.18), we obtain:

$a(u_{n,i}, v_i - u_{n,i}) \geq f(v_i - u_{n,i}), \forall v_i \in u_n + K_{n,i}.$

Taking $v_{n,i} = u_{n,i} - u_n$, we observe that $v_{n,i}$ provides a solution of the problem (2.12). This, together with the uniqueness of the solutions of the problems (2.15) and (2.12), goes to the wanted equivalence. By computing, we have:

$$a(u_n + v_{n,i}, v_i - v_{n,i}) - a(u_n + v_{n,i}, u_n) \geq f(v_i - v_{n,i}) - f(u_n).$$

From the demonstration of lemma 2.3 it results that the new approximation u_{n+1} lies in K as long as $u_n \in K$ and $\rho(x)N(x) \leq 1$. We assume that $u - u_n \in K$. We want to know if $u - u_{n+1} \in K$ is valid under the same constraint of ρ. The answer is positive. To see why this holds, we observe that from (2.13) we have:

$$u - u_{n+1} = u - u_n - \rho \sum_{i=1}^{m} v_{n,i}.$$

We observe that from lemma 2.4 we have:

$u_n + v_{n,i} = u_{n,i} \leq u \Rightarrow v_{n,i} \leq u - u_n.$

Therefore,

$$\sum_{i=1}^{m} v_{n,i}(x) \leq N(x)(u(x) - u_n(x)).$$

It results that

$$u - u_n - \rho \sum_{i=1}^{m} v_{n,i} \geq u - u_n - \rho N(x)(u - u_n) = (1 - \rho N)(u - u_n) \geq 0.$$

Thus, $u - u_{n+1} \in K.$

The result can be summarized as follows:

Theorem 2.3: Let u be the solution of the inequality (2.8) and let { u_n} be a sequence of approximations given by the algorithm 2.2, in which the parameter ρ is chosen according to the lemma 2.3. If the initial guess u_0 is selected such that u_0, $u - u_0 \in K$, then u_{n+1}, $u - u_{n+1} \in K$. Furthermore, the problem (2.12) is equivalent (2.15) in the sense that $u_{n,i} = u_n + u_{n,i}$.

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