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MAX-STABLE DISTRIBUTION– part I

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Abstract: *The topic is present, because more and more frequently we have been dealing with the issue of extreme values predictions which can be registered for certain phenomena which possess a random behaviour due to their very nature. The limitation of a distribution corresponding to a selection of identically distributed and independent random variables* (*v.a.i.i.d*) *having the stability property defines a distribution class called „max-stable*''.

Keywords: *random variables, distribution function, convergence, probability, sequences, maximumlikelihood estimation.*

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1. Introduction

May $\varepsilon_1, \varepsilon_2, ... \varepsilon_n$ a selection of independent and identically distributed random variables and $M_n = \max\{\varepsilon_1, \varepsilon_2, .. \varepsilon_n\}$, $m_n = \min\{\varepsilon_1, \varepsilon_2, .. \varepsilon_n\} = -\max\{-\varepsilon_1, -\varepsilon_2, ... -\varepsilon_n\}$.

We consider M_n's distribution function as F^n:

$$F^n(x) = P(M_n \le x) = \\ = P(\varepsilon_1 \le x; \varepsilon_2 \le x; ... \varepsilon_n \le x) \qquad (1.1)$$

where F is the common distribution function of ε_i, $i = \overline{1,n}$.

The possibilities for the distribution's limit define a class with a certain property of stability, made of the so-called max-stable distributions. This class is formed by three distributions with extreme values and each max-stable distribution G has one of the parametrical forms, defined as follows:

Type I: $G(x) = \exp(-e^{-x}),\ x \in R$,

Type II: $G(x) = \begin{cases} 0, x \le 0 \\ \exp\left(-x^{-\alpha}\right), \alpha > 0, x > 0 \end{cases}$,

Type III: $G(x) = \begin{cases} \exp\left(-(-x)^{\alpha}\right), \alpha > 0, x \le 0 \\ 1, \qquad x > 0 \end{cases}$.

The study refers especially to the conditions in which for the adequate normalizing constants $a_n > 0$, b_n, there should be verified that:

$$P\{a_n(M_n - b_n) \le x\} \xrightarrow{w} G(x) \qquad (1.2)$$

were $\xrightarrow{w}$ denotes „weak convergence"

It will be shown that the convergence appears in the continuity points of G, afterwards determining which G of the distribution function can appear as a limit and it is possible that G non confluent of the distribution function should form the max-stabile distribution class.

Using the relations (1.1) and (1.2) we obtain:

$$P\{a_n(M_n - b_n) \le x\} = \\ = F^n\left(a_n^{-1}x + b_n\right) \xrightarrow{w} G(x) \tag{1.3}$$

2. CONVERSE FUNCTIONS AND KHINTCHINE'S CONVERGENCE THEOREM

The inverse of the monotonous functions can be defined in different ways according to the intended purpose. For the following study, let us choose the next means for building these functions.

If Ψ is a continuous function, nondecreasing, we define the inverse function Ψ^{-1} on the interval $(\inf\{\Psi(x)\};\sup\{\Psi(x)\})$ through $\Psi^{-1}(y) = \inf\{x; \Psi(x) \ge y\}$.

The definition domain of the function Ψ^{-1} is presented as an open interval, but it can also be closed at any end $\inf\{\Psi(x)\}$ respectively $\sup\{\Psi(x)\}$ tempts towards finite values for *x*.

Lemma 2.1.

1) For Ψ previously defined , if $a > 0, b, c$ constant and $H(x) = \Psi(ax+b) - c$ then $H^{-1}(y) = a^{-1}\left(\Psi^{-1}(y+c) - b\right)$.

2) For Ψ previously defined, if Ψ^{-1} is continuous , then $\Psi^{-1}(\Psi(x)) = x$.

3) If G is a nondegenerate distribution function, then there exists $y_1 < y_2$ so that $G^{-1}(y_1) < G^{-1}(y_2)$ are well defined (and finite).

Proof

1) $H^{-1}(y) = \inf\{x; \Psi(ax+b) - c \ge y\}$

$= a^{-1}\left(\inf\{(ax+b); \Psi(ax+b) \ge y+c\} - b\right)$

$= a^{-1}\left(\Psi^{-1}(y+c) - b\right)$ Q.E.D.

2) According to the definition of Ψ^{-1} and property of function Ψ as nondecreasing, we obtain $\Psi^{-1}(\Psi(x)) \le x$.

If there is a z so that for $z < x$ we have $\Psi(z) \ge \Psi(x)$ (*) then taking into account that Ψ is nondecreasing, we get $\Psi(z) \le \Psi(x)$ (**) so, from relations (*) and (**) we obtain $\Psi(z) = \Psi(x)$.

For $y = \Psi(z) = \Psi(x)$ we have $\Psi^{-1}(y) \le z$, while for $y > \Psi(z) = \Psi(x)$ we have $\Psi^{-1}(y) \ge x$, which contradicts Ψ^{-1}'s continuity, therefore $\Psi^{-1}(\Psi(x)) = x$.

3) If G is nondecreasing, then there is $x_1^{'} < x_2^{'}$ so that $0 < G\left(x_1^{'}\right) = y_1 < G\left(x_2^{'}\right) = y_2 \le 1$. Obviously $G^{-1}(y_1) = x_1$ and $G^{-1}(y_2) = x_2$ are well defined. But $G^{-1}(y_2) \ge x_1^{'}$ and the equality imposes $G(z) \ge y_2$ for all $z \ge x_1$ so $G\left(x_1^{'}\right) = \lim\limits_{\varepsilon > 0} G\left(x_1^{'} + \varepsilon\right) \ge y_2$, as $G\left(x_1^{'}\right) = y_1$ there results contradiction, so $G^{-1}(y_2) > x_1^{'} \ge x_1 = G^{-1}(y_1)$ Q.E.D.

Corollary 2.2

If G is a nondecreasing distribution function and a, α, b, β constants with $a > 0, \alpha > 0$ so that $G(ax+b) = G(\alpha x + \beta)$ for any *x*, then $a = \alpha$ and $b = \beta$.

Proof

Let $y_1 < y_2$ and $-\infty < x_1 < x_2 < \infty$ which fulfill the request (3) from Lemma 2.1, such that $x_1 = G^{-1}(y_1)$ and $x_2 = G^{-1}(y_2)$. Using the request (1) from Lemma 2.1 we obtain $a^{-1}\left(G^{-1}(y) - b\right) = \alpha^{-1}\left(G^{-1}(y) - \beta\right)$, any *y*. For $y = y_1$ respectively $y = y_2$ we obtain :

$$\begin{cases} a^{-1}(x_1 - b) = \alpha^{-1}(x_1 - \beta) \\ a^{-1}(x_2 - b) = \alpha^{-1}(x_2 - \beta) \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} a^{-1} = \alpha^{-1} \\ -a^{-1} \cdot b = -\alpha^{-1} \cdot \beta \end{cases} \Rightarrow \begin{cases} a = \alpha \\ b = \beta \end{cases}$$

Theorem (Khintchine) (2.3)

Let the system $\{F_n\}$ of distribution functions and let G be a nondecreasing distribution function. We consider the adequate constants $a_n > 0, b_n$ so that:

$$F_n(a_n x + b_n) \xrightarrow{w} G(x) \tag{2.1}$$

Then for certain $G_\bullet$ nondecreasing distributive functions and for the constants $\alpha_n > 0, \beta_n$ there takes place the relation:

$$F_n(\alpha_n x + \beta_n) \xrightarrow{w} G_\bullet(x) \qquad (2.2)$$

if and only if

$$a_n^{-1}\alpha_n \to a \text{ and } a_n^{-1}(\beta_n - b_n) \to b \qquad (2.3)$$

For each $a > 0$ and b there takes place the equality :

$$G_\bullet(x) = G(ax + b) \qquad (2.4)$$

Proof

„$\Leftarrow$" Suppose first (2.3) is true and proove (2.2).

For $a_n > 0, b_n$ we note $F_n(a_n x + b_n) = F_n'(x)$ so according to (2.1) we obtain:

$$F_n'(x) \xrightarrow{w} G(x) \qquad (2.1')$$

We note $a_n^{-1}\alpha_n = \alpha_n'$ and $a_n^{-1}(\beta_n - b_n) = \beta_n'$ and we get:

$$F_n'(\alpha_n' x + \beta_n') \xrightarrow{w} G_\bullet(x) \qquad (2.2')$$

$$\left.\begin{array}{l}\alpha_n' \to a,\ a > 0 \\ \beta_n' \to b\end{array}\right\} \qquad (2.3')$$

If (2.1'), (2.3'), (2.4) are true then:

$$F_n'(\alpha_n' x + \beta_n') \xrightarrow{w} G(ax + b) = G_\bullet(x)$$

That is $F_n(\alpha_n x + \beta_n) \to G_\bullet(x)$.

„$\Rightarrow$" Suppose first (2.2) is true and proove (2.3).

As $G_\bullet$ is a nondecreasing distribution function there is x' and x'' so that $0 < G_\bullet(x') < 1$ and $0 < G_\bullet(x'') < 1$.

We consider the sequences $\{\alpha_n' x' + \beta_n'\}$ and $\{\alpha_n' x'' + \beta_n'\}$ which must be bounded .

We assume that $\{\alpha_n' x' + \beta_n'\}$ is not bounded then we choose a subsequence $\{\alpha_{n_k}' x' + \beta_{n_k}'\}$ convergent to $+\infty$ and according to the relation (2.1') we obtain:

$$F_{n_k}'\left(\alpha_{n_k}' x' + \beta_{n_k}'\right) \xrightarrow[x = x']{\to 0 \text{ or } 1} G_\bullet(x') \in (0,1) \Rightarrow$$

contradiction so the sequence $\{\alpha_n' x' + \beta_n'\}$ is bounded. Analog for $\{\alpha_n' x'' + \beta_n'\}$. If $\{\alpha_n' x' + \beta_n'\}$ and $\{\alpha_n' x'' + \beta_n'\}$ are bounded then $\{\alpha_n'\}, \{\beta_n'\}$ are bounded.

If the subsequences $\{\alpha_{n_k}'\}, \{\beta_{n_k}'\}$ are convergent then there are a, b so that $\alpha_{n_k}' \to a$, $\beta_{n_k}' \to b$ so

$$F_{n_k}'(\alpha_{n_k}' x' + \beta_{n_k}') \to G(ax + b) = G_\bullet(x),$$

$G_\bullet$ being a nondecreasing distribution function with $a > 0$.

If we consider another subsequence $\{\alpha_{n_r}' x' + \beta_{n_r}'\}$ bounded with $\alpha_{n_r}' \to a'$ and $\beta_{n_r}' \to b'$ then:

$$F_{n_r}'(\alpha_{n_r}' x' + \beta_{n_r}') \to G(a'x + b') = G_\bullet(x).$$

According to the corollary 2.2 $G(ax + b) = G_\bullet(x) = G(a'x + b')$ implies $a = a'$ and $b = b'$ Q.E.D.

3. Max – stable distributions

In this part we look for those G of the distribution functions which are possible asimpthotic laws for the maximum of the distributed independent and identical selections which form the max–stable distributions class.

Definition 3.1. G, a nondecreasing distribution function, is called max–stable if there are the constants a_n, b_n so that

$G^n(a_n x + b_n) = G(x)$, for each n = 2,3,..., for all $x \in \mathrm{R}$. (3.1)

Theorem 3.1.

a) G nondecreasing distribution function is max–stable if and only if there is a sequence $\{F_n\}$ of the distribution functions and the constants $a_n > 0,\ b_n$ so that

$$F_n\left(a_{nk}^{-1} x + b_{nk}\right) \xrightarrow{w} G^{\frac{1}{k}}(x) \text{ for } n \to \infty$$

for each $k = 1,2,...$ (3.2)

b) If G is nondecreasing, $D(G)$ is non void if and only if G is max–stable. At the same time $G \in D(G)$

Proof

a) If G is nondecreasing, so there is $G^{1/k}$ for each k and if (3.2) is true for each k, Theorem 2.3 (with a_n^{-1} for a_n) implies $G^{1/k}(x) = G(\alpha_k x + \beta_k)$, for some $a_k > 0$ and β_k, so that G is max-stable. Conversely, if G is max-stable and $F_n = G^n$ we have $G^n\left(a_{nk}^{-1} x + b_{nk}\right) = G(x)$ for some $a_n > 0$, b_n and

$$F_n\left(a_{nk}^{-1} x + b_{nk}\right) = \left(G^{nk}\left(a_{nk}^{-1} x + b_{nk}\right)\right)^{1/k} = \\ = (G(x))^{1/k}$$

so that (3.2) becomes evident.

b) If G is max-stable $G^n\left(a_n x + b_n\right) = D(x)$ $a_n > 0$, b_n (therefore $n \to \infty$) we see that $G \in D(G)$. Conversely, if $D(G)$ is non void $F \in D(G)$ let us say that $F^n\left(a_n^{-1} x + b_n\right) \xrightarrow{w} G(x)$. Consequently $F^{nk}\left(a_{nk}^{-1} x + b_{nk}\right) \xrightarrow{w} G(x)$ or $F^n\left(a_{nk}^{-1} x + b_{nk}\right) \xrightarrow{w} G^{1/k}(x)$. Therefore (3.2) is true for $F_n = F^n$ and consequently from a) G is max-stable.

Corollary 3.2

If G is max–stable, there are real functions $a(s) > 0, b(s) > 0$ defined for $s > 0$ so that :

$G^s\left(a(s)x + b(s)\right) = G(x),\ x, s > 0$ (3.3)

Proof

As G is max–stable, there is $a_n > 0,\ b_n$ so that

$G^n\left(a_n x + b_n\right) = G(x)$ (3.4)

Then $G^{[ns]}(a_{[ns]} x + b_{[ns]}) = G(x)$, but

$G^n(a_{[ns]} x + b_{[ns]}) \xrightarrow{w} G^{\frac{1}{s}}(x)$ (3.5)

From the relations (3.5) and (3.4) and as $G^{\frac{1}{s}}$ is nondecreasing, theorem 2.3 is applied with $\alpha_n = a_{[ns]}$ and $\beta_n = b_{[ns]}$ to show that $G\left(a(s)x + b(s)\right) = G^{\frac{1}{s}}(x)$ for some $a(s) > 0$ and $b(s)$, as requested.

Definition 3.2.

We can say that two distribution function G_1, G_2 are of the same type if $G_2(x) = G_1(ax + b)$, for some constants $a > 0,\ b$. Then the above definition of the max–stabile distribution can be rephrased as follows: A distribution function G nondecreasing is max–stable if for each $n = 2,3,...$ G^n distribution function is of the same type as G.

Further the theorem 2.3 shows that if $\{F_n\}$ is a selection of distribution function with $F_n(a_n x + b_n) \xrightarrow{w} G_1$, $F_n(\alpha_n x + \beta_n) \xrightarrow{w} G_2$, $a_n > 0, \alpha_n > 0$, then G_1, G_2 are of the same type, taking into account that they are non decreasing. So distribution function can be divided into equivalent classes (that we call types) saying that G_1 and G_2 are equivalent if $G_2(x) = G_1(ax + b)$ for some $a > 0,\ b$.

If G_1 and G_2 are distribution function of the same type $(G_2(x) = G_1(ax + b))$ and $F \in D(G_1)$ so that $F^n(a_n x + b_n) \xrightarrow{w} G_1$, for some $a_n > 0, b_n$, then the relation (2.3) is fulfilled with $\alpha_n = a_n a,\ \beta_n = b_n + a_n b$ so that

$F^n(\alpha_n x+\beta_n)\xrightarrow{w} G_2(x)$ by theorem 2.3 and consequently $F\in D(G_2)$.

Thus if G_1, G_2 are of the same type $D(G_1)=D(G_2)$. Similarly we can see the theorem from theorem 2.3 that if $F\in D(G_1)$ and $F\in D(G_2)$ then G_1 and G_2 of the same type . from this reason $D(G_1), D(G_2)$ are identical if G_1 and G_2 are of the same type and are thus disjoint .

That means that the attraction domain of G distribution function depends only on G' s type.

Theorem 3.3

Each max–stable distribution is of the same type as one of the following three distributions $G(ax+b)$ for a $a>0$, b some for

Type I: $G(x)=\exp(-e^{-x}), -\infty<x<\infty$

$$\text{Type II}: G(x)=\begin{cases}0 & x\le 0\\ \exp(-x^{-\alpha}),\ \alpha>0 & x>0\end{cases}$$

$$\text{Type III}: G(x)=\begin{cases}\exp(-(-x^{-\alpha})),\ \alpha>0 & x\le 0\\ 1 & x>0\end{cases}$$

Reciprocally, each distribution of the type of extreme value is max–stable.

Proof

From this moment, the centering is clear on the example for type I.

$$\left(\exp\left\{-e^{-(ax+b)}\right\}\right)^n=\exp\left\{-e^{-(ax+b-\ln n)}\right\}$$

with similar expressions for types II and III.

If G is max-stable, then (3.3) is valid for all $s>0$ and all x.

If $0<G(x)<1$, (3.2) gives $s(-\log G(a(s)x+b(s)))=-\log G(x)$, so that

$$-\log(-\log G(a(s)x+b(s)))-\log s=$$
$$=-\log(-\log G(x)).$$

Now we can notice from the max–stable property with $n=2$ that G can not tempt to any finite point. Thus, for the nondecreasing function $\Psi(x)=-\log(-\log G(x))$ we have $\inf\{\Psi(x)\}=-\infty, \sup\{\Psi(x)\}=+\infty$ and has as inverse function the function $U(y)$ defined for all $y\in \mathrm{R}$.

Furthermore

$\Psi(a(s)x+b(s))-\log s=\Psi(x)$, so that from lemma 2.1(1)

$$\frac{U(y+\log s)-b(s)}{a(s)}=U(y).$$

For $y=0\Rightarrow U(0)=\dfrac{U(\log s)-b(s)}{a(s)}\Rightarrow$

$$\frac{U(y+\log s)-U(\log s)}{a(s)}=U(y)-U(0)$$ and

writing $z=\log s, a(z)=a(e^z)$ and $U(y)=U(y)-U(0)$.

For all $y,z\in \mathrm{R}$

$$U(y+z)-U(z)=U(y)a(z) \qquad (3.6)$$

Exchanging y with z we obtain:

$$\tilde{U}(z+y)-\tilde{U}(y)=\tilde{U}(z)\tilde{a}(z) \qquad (3.6')$$

From (3.6) and (3.6') it results $\tilde{U}(y+z)=\tilde{U}(z+y)\Rightarrow$

$$U(y)(1-a(z))=U(z)(1-a(y)) \qquad (3.7)$$

There are two possible cases (a), (b) as follows:

(a) $a(z)=1$ for all z we have from (3.6)

$$U(y+z)=U(y)+U(z)$$

The only monotonous increasing solution is $U=\rho y$ for $\rho>0$, so that

$U(y)-U(0)=\rho y$ or

$\Psi^{-1}(y)=U(y)=\rho y+\nu$, $\nu=U(0)$.

As Ψ is continuous the lemma 2.1(2) gives

$x=\Psi^{-1}(\Psi(x))=\rho\Psi+\nu \Rightarrow x=\rho\Psi(x)+\nu\Rightarrow$

$$\begin{cases}\Psi(x)=\dfrac{x-\nu}{\rho} \\ \Psi(x)=-\log(-\log G(x))\end{cases}\Rightarrow$$

$$\Rightarrow -\log(-\log G(x))=\frac{x-\nu}{\rho}$$

$$\log(-\log G(x))=-\frac{x-\nu}{\rho}\Rightarrow$$

$$\Rightarrow -\log G(x)=e^{-\frac{x-\nu}{\rho}}=\exp\left(-\frac{x-\nu}{\rho}\right)$$

$$\log G(x)=-\exp\left(-\frac{x-\nu}{\rho}\right)\Rightarrow$$

$$\Rightarrow G(x)=\exp\left(-\exp\left(-\frac{x-\nu}{\rho}\right)\right)$$

$$G(x)=\exp\{-e^{-(x-\nu)/\rho}\},\ 0<G(x)<1$$

G can not reach any finite point and consequently it has the above form for all x and thus it is of type I.

(b) $a(z)\neq 1$ for a z the relation (3.7) gives:

$$U(y)=\frac{U(z)}{1-a(z)}(1-a(y))=c(1-a(y)) \quad (3.8)$$

Where $c=\dfrac{U(z)}{1-a(z)}\neq 0$, ($U(z)=0$ implies $U(y)=0$ for all y and consequently $U(y)=U(0)$ is constant).

From (3.6) we obtain as following:

$$c(1-a(y+z))-c(1-a(z))=c(1-a(y))a(z)$$

which gives $a(y+z)=a(y)a(z)$. But a is monotonous (from 3.8) and the only non constant solutions of the functional equation have the form $a(y)=e^{\rho y}$ for $\rho\neq 0$. Thus :

$$\Psi^{-1}(y)=U(y)=\nu+c(1-e^{\rho y})$$

where $y=U(0)$. As $-\log(-\log G(x))$ is increasing , the same for U, thus we must have $c<0$ if $\rho>0$ and $c>0$ if $\rho<0$.

From lemma 2.1(2)

$$x=\Psi^{-1}(\Psi(x))=\nu+c(1-e^{\rho\Psi(x)})=$$
$$=\nu+c(1-(-\log G(x))^{-\rho}),$$

$0<G(x)<1$:

$$G(x)=\exp\left\{-\left(1-\frac{x-\nu}{c}\right)^{-1/\rho}\right\}$$

From its continuity in any finite point we can see that G is of type II or type III, with $a=+1/\rho$ or $a=-1/\rho$ as , if $\rho>0, c<0$ and $\rho<0$ if $c>0$.

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