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COLORING OF THE SIMPLICIAL COMPLEX AND THE GRUNDY FUNCTION

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Abstract: *The coloring problem of all elements of the complex of multi-ary relation is formulated. We give a necessary and sufficient conditions for the existence of a correct coloring of this complex. The Grundy function role to solving this problem is studied.*

Keywords: *complex of multi-ary relations, Grundy function, simplex, p-cromatic complex, coloring problem, graph, hypergraph, n-dimensional loop.*

Let $X = \{x_1, x_2, ..., x_r\}$ be a finite set of elements and $X = X^1, ..., X^{n+1}, ...,$ $(n \geq 1)$ be the succession of Cartesian products $X : X^{m+1} = X^m \cdot X,$ $1 \leq m \leq n.$ Any nonempty subset $R^m \subset X^m$ is said to be an *m-ary relation* of elements from X (the set $R^1 \subset X^1$ is a subset of elements from X).

According to the mentioned above, an *m-ary relation* R^m is a family of ordered successions named sequences. Each sequence consists of m elements of X . Generally speaking, the sequence $(x_{i_1}, x_{i_2}, ..., x_{i_m}) \in R^m$ could contain some elements from X several times. For this kind of sequence any subsequence $(x_{j_1}, x_{j_2}, ..., x_{j_l}),$ $1 \leq l \leq m,$ which preserves the order of elements of $(x_{i_1}, x_{i_2}, ..., x_{i_m})$ is called a *hereditary subsequence*.

Definition 1. *A finite family of relations* $\{R^1, R^2, ..., R^{n+1}\}$ *which satisfies the conditions:*

1. $R^1 = X^1 = X;$

2. $R^{n+1} \neq \emptyset;$

3. any hereditary subsequence $(x_{j_1}, x_{j_2}, ..., x_{j_l}),$ $1 \leq l \leq m \leq n+1,$ *of the sequence* $(x_{i_1}, x_{i_2}, ..., x_{i_m}) \in R^m$ *belongs to the l-ary relation* R^l *,*

is called a ***generalized complex of multi-ary relations (*** G ***-complex)*** *and is denoted by*

$$G^{n+1} = (R^1, R^2, ..., R^{n+1}).$$

From Definition 1 we obtain that the set R^m of a generalized complex R^{n+1} is not empty for each $1 \leq m \leq n+1$.

The study of generalized complex of multi-ary relations is interesting because this notion covers a lot of classical notions like graphs and hypergraphs [2], matroids [4], simplicial complexes, etc. Certainly, this complex could be interpreted as a particular case of the abstract cellular **complex** (CW), but these new mathematical structures serve as effective models for solving a lot of theoretical and applicative problems. Remark that the object R^{n+1} has advantage over the structures mentioned above. Thus, if it is compared with cellular complex (CW) then it is seen that the R^{n+1} is formed from the elementary "bricks", maybe with non-isomorphic deformations, like there are the abstract quasi-simplexes.

Consider the generalized complex of relations $G^2 = (R^1, R^2)$. It is obvious that this complex represents a directed graph [5]. This allows us to consider the generalized complex of relations R^{n+1} as an **oriented** and **hereditary hypergraph** (according to C. Berge). The last notion could be rarely found in the bibliography of speciality, and it represents a structure different from the notion of hypergraph [3]. Next, we will describe a procedure that allows to obtain the notion of hypergraph in the form of generalized complex and the so-called cycles of hypergraph in a natural form. This will be different from the known one, and it will be starting from the notion of oriented hypergraph transformed into a complex of abstract simplexes.

Most about the G-complex of multi-ary relations see in [6, 7].

By analogy to the know classical bibliography in the combinatorial topology and topological algebra fields [1, 3], further we will also use other notations and notions, that are equivalent to those mentioned. These notations and notions will be used to study the properties of the complex of multi-ary relations, which are needed to solve practical problems.

Definition 2. *The sequence* $(x_{i_0}, x_{i_1}, ..., x_{i_m}) \in R^{m+1}$, *which has pairwise distinct elements, is said to be* ***an abstract simplex of dimension m*** *and donoted by* $S_i^m = (x_{i_0}, x_{i_0}, ..., x_{i_m}) \in R^{m+1}, m = \dim S_i^m$.
Any sequence of elements $(x_{j_0}, x_{j_1}, ..., x_{j_t}) \in R^{l+1}$, *which is a hereditary subsequence of elements* S_i^m, *is called a* ***face*** *of dimension l of a simplex* S_i^m, *and it will be denoted by* $S_j^l = (x_{j_0}, x_{j_1}, ..., x_{j_t})$, $S_j^l \subset S_i^m$. *Sometimes we will be denoted by of dimension zero –* ***vertices,*** *and those of dimension one are called* ***edges*** *of simplex* $S_i^m, 0 \le m \le n$.

A subset formed by m + 1 pairwise distinct elements from the set X can generate several abstract simplexes of dimension m. The maximal number of it coincides with the number of different permutations of the m + 1 elements. This means that there are (m + 1)! simplexes. It follows that distinct abstract simplexes of dimension m that are stretched on m + 1 vertices from X could be imagined as membranes that strain these vertices.

Further we will denote by S^m the set of all simplexes with dimension m that are determined by sequences from R^{m+1}.

By this way the complex of relations $R^{n+1} = (R^1, R^2, ..., R^{n+1})$ can represented as follows: $S^0 = R^1, S^1 = R^2, ..., S^n = R^{n+1}$ and $(S^0, S^1, ..., S^n) = K^n$.

Thus, we believe that K^n is an abstract simplicial complex, also.

Definition 3. *An abstract simplicial subcomplex*

$$K^m = S^0 \cup S^1 \cup ... \cup S^m \qquad (1)$$

of K^n *is said to be m-dimensional skeleton of* K^n, $m = 0, 1, ..., n$.

We will consider a multi-valued operator $I : K^n \to K^n$ with the property that for each $S_i^m \in K$ the equality $I(S_i^m) = K(S_i^m)$, holds, where $K(S_i^m)$ represents the set of simplices of the complex K^n, too.

Definition 4. *Given a complex* K^n, *the set of nonnegative integers* N_0 *and a single valued mapping*

$$g : K^n \to N_0.$$

The mapping g *is called the Grundy function of the complex* K^n *if for any* $S_i^m \in K^n$ *the following equality holds:*

$$g(S_i^m) = \min_{S_i^m \in K^n}\{N_0 \setminus g(I(S_i^m))\},$$

$i = 1, 2, ..., t_m$, $m = 0, 1, ..., n$, where $g(I(S_i^m))$ denotes the set of values of g for all simpleces from $I(S_i^m)$.

Definition 5. *The mapping*

$$\Gamma_n : K^n \to N_0 \qquad (2)$$

is said to be a color function of the complex K^n if Γ_n is a single-valued mapping and for each two simplices $S_i^{m_1}$, $S_j^{m_2} \in K^n$, which are different and have non-empty intersection or which are not neigbouring for $m_1 = m_2 = 0$ the following inequality holds:

$$\Gamma_n(S_i^{m_1}) \neq \Gamma_n(S_j^{m_2}) \qquad (3)$$

In this case we will say that the family of simplices from K^n has a proper coloring. Let $\overline{0, \gamma_n - 1} \in N_0$ be the interval of minimal length for which there exists a proper coloring of the elements of K^n, i.e., for every $S_i^{m_1} \in K^n$ the value $\Gamma_n(S_i^{m_1})$ satisfies the relation $\Gamma_n(S_i^{m_1}) \leq \gamma_n - 1$ and there exists a simplex $S_j^{m_2} \in K^n$ such that $\Gamma_n(S_j^{m_2}) = \gamma_n - 1$. The number γ_n is called n-dimensional chromatic number of the complex K^n.

Considering the restriction (2) for the skeleton (1), i.e.,

$$\Gamma_m : K^m \to N_0, \qquad (4)$$

we will analogously speak about the *m-dimensional chromatic number*, γ_m, of the complex K^n, $m = 0, 1, 2, ..., n-1$. Let $\gamma_0 = 1$ for $m = 0$. The complex K^n is called *p-chromatic* if there exists a coloring of elements of K^n such that (2) satisfies (3) and can be written:

$$\Gamma_n : K^n \to \overline{0, p-1},$$

where for $\forall q \in \overline{0, p-1}$ exists $S \in K^n$ such that $\Gamma_n(S) = q$.

Now it is obvious that there exists a number $p \in N_0$ for which the complex K^n is *p*-chromatic. For this it is sufficient to enumerate all elements of K^n. It is not an easy problem to find a minimal number *p*.

Theorem 1. *Given an abstract simplicial p-chromatic complex K^n and the operator Γ_n satisfying (3). The number p equals γ_n if and only if there exists a Grundy function satisfying the equality:*

$$\max_{S \in K^n} g(S) = \gamma_n - 1. \qquad (5)$$

Proof. $\Rightarrow$ Let the equality (5) holds. We will show that in this case all elements of the complex K^n can be properly colored with the colors $0, 1, ..., \gamma_n - 1$. Denote by S_m the set of all simplices S with $g(S) = m$, $m = 0, 1, ..., \gamma_n - 1$. We will prove that each two simplices $S_1, S_2 \in S_m$ either have non-empty intersection or, if S_1 and S_2 are 0-dimensional simplicies, do not belong to some *k*-dimensional simplex of K^n with $k > 0$. Assume that $S_1 \cap S_2 = \varnothing$, or S_1, S_2 are 0-dimensional simplicies belonging to some 1-dimensional simplex of K^n. Then, e.g., for $g(S_1) = \min\{N_0 \setminus g(I(S_1))\}$ and according to the definition of a subcomplex, the relation $K(S_1) = I(S_1)$ holds, where $S_2 \in I(S_1)$. This contradicts the definition of the Grundy function. Thus, the necessity is proved.

$\Leftarrow$ Let there exists a proper coloring of K^n with γ_n colors: $0, 1, ..., \gamma_n - 1$, and $S_0, S_1, ..., S_{\gamma_{n-1}}$ be the set of families of all simplices of K^n colored with colors $0, 1, \ldots, \gamma_n - 1$, respectively. Our aim is to construct another set $\overline{S_0}, \overline{S_1}, \ldots, \overline{S_{\gamma_n - 1}}$ and to define for this set the function $g_m(S) = m \Leftrightarrow S \in \overline{S_m}$, $m = 0, 1, \ldots, \gamma_n - 1$.

0) By induction the set $\overline{S_0}, \ldots, \overline{S_{\gamma_n - 1}}$ can be constructed in the following way: $S_0^i = S_0^{i-1} \bigcup S_i^1$, where $S_i^1 = S_i \setminus I(S_0^{i-1})$, for all $1 \le i \le \gamma_n - 1$. (here we consider $S_0^0 = S_0$).

Consider $\overline{S^0} = S_0^{\gamma_n - 1}$.

Form another set:

$$^{1}S_1,\ ^{1}S_2, \ldots,\ ^{1}S_{\gamma_n - 1}, \qquad (6)$$

where $^{1}S_m = S_m \setminus \overline{S_0}$, $m = 1, 2, \ldots, \gamma_{n-1}$.

The set (6) has the proprety: $^{1}S_m \ne \varnothing$, $m = 1, 2, \ldots, \gamma_n - 1$. Otherwise, if there exists an m_0 such that $^{1}S_{m_0} = \varnothing$, then the complex K^n can be colored with less than colors. This is possible due to the construction of the se $\overline{S_0},\ ^{1}S_1, \ldots,\ ^{1}S_{\gamma_{n-1}}$ and property of the multi-valued operator $I : K^n \to K^n$.

m-1) Suppose that the following set is constructed:

$$^{m-1}S_{m-1},\ ^{m-1}S_m, \ldots,\ ^{m-1}S_{\gamma_n - 1}, \quad (7)$$

where no family of simplices from (7) is empty. Consider the family $\overline{S_{m-1}} = {}^{m-1}S_{m-1}^{\gamma_n - 1}$ and construct the set:

$$^{m-1}S_{m-1},\ ^{m-1}S_m, \ldots,\ ^{m}S_{\gamma_n - 1}, \qquad (8)$$

where

$$^{m}S_m = S_m \setminus \overline{S_{m-1}},\ ^{m}S_{m+1} = S_{m+1} \setminus \overline{S_{m-1}}, \ldots,$$
$$^{m}S_{\gamma_{n-1}} = S_{\gamma_n - 1} \setminus \overline{S_{m-1}}.$$

m) Let $^{m}S_m^i = {}^{m}S_m^{i-1} \bigcup {}^{m}S_i^1$,

where $^{m}S_i^1 = {}^{m}S_i \setminus I({}^{m}S_m^{i-1})$, for all

$$m + 1 \le i \le \gamma_n - 1.$$

Consider $\overline{S_m} = {}^{m}S_m^{\gamma_n - 1}$ and construct the following set:

$$^{m+1}S_{m+1},\ ^{m+1}S_{m+2}, \ldots,\ ^{m+1}S_{\gamma_n - 1}, \qquad (9)$$

where

$$^{m+1}S_{m+i} = {}^{m+1}S_{m+1} \setminus \overline{S_m},\ i = 1, 2, \ldots, \gamma_n - 1 - m.$$

By construction the set $\overline{S_0}, \ldots, \overline{S_{\gamma_n - 1}}$ is obtained, having the property $\overline{S_m} \ne \varnothing$, $m = 0, 1, \ldots, \gamma_n - 1$. We will show that $g_m(S) = m$ is a Grundy function i.e.,

$$m = g(S) = \min\{N_0 \setminus g(I(S))\}, \qquad (10)$$

where $S \in \overline{S_m}$, $m = 0, 1, \ldots, \gamma_n - 1$.

The relation (10) is obvious (by construction of family $\overline{S_0}, \overline{S_1}, \ldots, \overline{S_m}, \ldots, \overline{S_{\gamma_n - 1}}$). Ineed, if $S \in S_m$, then the following holds:

$$I(S) \bigcap \overline{S_0} \ne \varnothing, \ldots, I(S) \bigcap \overline{S_{m-1}} \ne \varnothing,$$
$$I(S) \bigcap \overline{S_m} \ne \varnothing,$$
$$I(S) \bigcap \overline{S_{m+1}} \ne \varnothing, \ldots, I(S) \bigcap \overline{S_{\gamma_n - 1}} \ne \varnothing.$$

The terms of Definition 4 are satisfied: $m = g(S) = \min\{N_0 \setminus g(I(S))\}$, i.e., (10) holds.

Theorem 2. *Given an abstract simplicial complex K^n and the operator (2) with the property (3). The complex K^n is p-chromatic if and only if there exists a Grundy function of K^n satisfying inequality* $\max\limits_{S \in K^n} g(S) \le p - 1$.

The proof of this theorem is almost the same as that of Theorem 1.

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